Exercise 1

ACME manufacturing has 13 staff. Eight are hired by ACME directly, whereas five are independent contractors working for ACME. Two staff are on sick leave. We do not know whether they are ACME's own employees or contractors, but we do know that all staff have the same probability to be ill. ACME allocates to a new task the first staff that arrives at their premises in the morning (the order of arrival is random).

1) Call A the number of ACME's <u>own</u> employees that are on sick leave. Compute the PMF of A.
2) Call B the number of ACME's <u>own</u> employees that are allocated to the new task. Compute the PMF of B.

Exercise 2

Consider a computing system that accepts a workload consisting of exponential requests, arriving at a rate λ. The system serves requests using one server, whose rate is μ. However, when the number of jobs in the system exceeds a threshold $K \geq 1$, a *second* server is activated.

1) Model the system and draw its CTMC
2) Identify the stability condition and state explicitly its dependence on K.
3) compute the steady-state probabilities as a function of K.
4) Compute the probability that an arriving job finds exactly K jobs in the system, and compute under what conditions K is the most likely number of jobs that an arriving job will see. Justify the result.
5) Compute the average number of jobs in the system when $\lambda = \mu$, as a function of K. Justify the result.

Exercise 1 - Solution

1) RV A can take values 0, 1, 2. We are in a UPM, hence the PMF is given by the following expression:

$$P\{A = j\} = \frac{\binom{8}{j}\binom{5}{2-j}}{\binom{13}{2}}.$$

More specifically, it is $P\{A = 0\} = 5/39 \cong 0.128$, $P\{A = 1\} = 20/39 \cong 0.513$, $P\{A = 2\} = 14/39 \cong 0.359$

2) The PDF of RV B can be computed using total probability, by conditioning to A. In fact, it is:

$$P\{B = b|A = k\} = \frac{\binom{8-k}{b}\binom{(13-2)-(8-k)}{1-b}}{\binom{13-2}{1}}.$$

Once the above has been computed with $b \in \{0,1\}$ and $k \in \{0,1,2\}$, the following expression computes the PDF of B:

$$P\{B = b\} = \sum_{k=0}^{2} P\{B = b|Y = k\} \cdot P\{Y = k\}$$

The numerators of the above conditional probabilities are in the table below (the denominator is 11):

		b	
		0	**1**
k	**0**	3	8
	1	4	7
	2	5	6

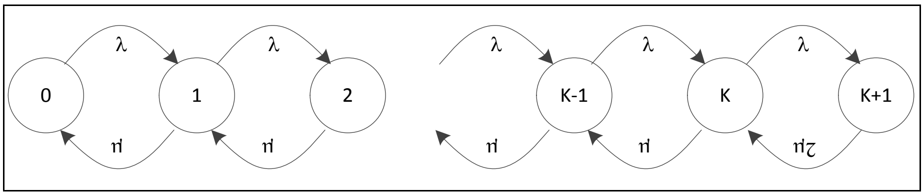
We obtain:

$$P\{B = 0\} = \frac{3}{11} \cdot \frac{5}{39} + \frac{4}{11} \cdot \frac{20}{39} + \frac{5}{11} \cdot \frac{14}{39} = \frac{165}{11 \cdot 39} = \frac{15}{39} \cong 0.384$$

$$P\{B = 1\} = \frac{24}{39} \cong 0.615$$

Exercise 2 - Solution

1) The CTMC is as follows:



2) The stability condition can be easily inferred to be *independent of K*. In fact, the fact that the service rate is μ up to state *K* is irrelevant, as long as the remaining states (up to infinity) have a different service rate 2μ. This said, the stability condition can only be $\boxed{\lambda < 2\mu}$. The same result should emerge from the computations as well. In fact, the local balance equations yield:

$$p_i = \begin{cases} \frac{\lambda}{\mu} p_{i-1} & i \le K \\ \frac{\lambda}{2\mu} p_{i-1} & i > K \end{cases}, \text{ hence } p_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \cdot p_0 & i \le K \\ \left(\frac{\lambda}{2\mu}\right)^{i-K} \cdot \left(\frac{\lambda}{\mu}\right)^K \cdot p_0 & i > K \end{cases}$$

Therefore, the normalization condition is as follows:

$$p_0 \cdot \left[\sum_{i=0}^{K} \left(\frac{\lambda}{\mu}\right)^i + \sum_{i=K+1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^K \cdot \left(\frac{\lambda}{2\mu}\right)^{i-K} \right] = 1$$

Which can be rewritten as:

$$p_0 \cdot \left[\sum_{i=0}^{K} \left(\frac{\lambda}{\mu}\right)^i + \left(\frac{\lambda}{\mu}\right)^K \cdot \sum_{j=1}^{+\infty} \left(\frac{\lambda}{2\mu}\right)^j \right] = 1.$$

The first sum is always finite, and the second one converges if and only if $\boxed{\lambda < 2\mu}$. Call $u = \lambda/2\mu$.

3) We need to treat separately the two cases $\lambda = \mu, \lambda \ne \mu$ (because of the first sum).

Case 1: $\lambda = \mu$

In this case, we have $p_0 \cdot \left[(K+1) + \frac{u}{1-u} \right] = 1$, hence:

$$p_0 = \frac{1-u}{K \cdot (1-u) + 1}, \boxed{p_i = \begin{cases} p_0 & i \le K \\ u^{i-K} \cdot p_0 & i > K \end{cases}}$$

In this case, all s.s. probabilities are equal up to state *K*. This should not surprise us, since the states at the leftmost part of the diagram have equal transition rates to the left and to the right.

Case 2: $\lambda \neq \mu$

We get: $p_0 \cdot \left[\frac{1-(2u)^{K+1}}{1-2u} + (2u)^K \cdot \frac{u}{1-u} \right] = 1$, i.e. $p_0 = \frac{1}{\frac{1-(2u)^{K+1}}{1-2u} + \frac{(2u)^K \cdot u}{1-u}}$

The above expression can be simplified to $p_0 = \frac{(1-2u)\cdot(1-u)}{(1-u)-u\cdot(2u)^K}$.

$$\boxed{p_i = \begin{cases} (2u)^i \cdot p_0 & i \le K \\ u^{i-K} \cdot (2u)^K \cdot p_0 & i > K \end{cases}}.$$

4) This system has constant arrival rates, hence it possesses the PASTA property, i.e., $\boxed{r_K = p_K}$. The answer is, thus, to find the condition under which $p_K > p_i, \quad i \neq K$. Note that, if $\lambda = \mu$, the above condition is false (all states up to K have the same probability), therefore we can safely assume $\lambda \neq \mu$.

In this case, we observe that $u^{i-K} \cdot (2u)^K \cdot p_0$ is by definition a decreasing sequence (otherwise the system would not be stable). On the other hand, $(2u)^i \cdot p_0$ is an increasing sequence if $\mu < \lambda < 2\mu$. Therefore, state K will be the most likely number of jobs if and only if $\boxed{\mu < \lambda < 2\mu}$.

5) When $\lambda = \mu$ we have:

$$p_0 = \frac{1-u}{K\cdot(1-u)+1}, \quad p_i = \begin{cases} p_0 & i \le K \\ u^{i-K} \cdot p_0 & i > K \end{cases}$$

Hence,

$$\begin{aligned} E[N] &= \sum_{i=0}^{K} i \cdot p_0 + \sum_{i=K+1}^{+\infty} i \cdot u^{i-K} \cdot p_0 \\ &= p_0 \cdot \left[\sum_{i=0}^{K} i + \sum_{j=1}^{+\infty} (K+j) \cdot u^j \right] \\ &= p_0 \cdot \left[\frac{K\cdot(K+1)}{2} + K \cdot \frac{u}{1-u} + \frac{u}{(1-u)^2} \right] \end{aligned}$$

By substituting $u = 1/2$ we get

$$E[N] = \frac{1}{K+2} \cdot \left[\frac{K\cdot(K+1)}{2} + (K+2) \right] = \frac{K}{2} \cdot \frac{K+1}{K+2} + 1$$

The result makes sense. In fact, if $u = 1/2$, all the states up to *K* are equally likely (hence the first term), but states *above K* have non null probability. Therefore, the result must be larger than *K*/2, which would be the result if those state did not exist.