Exercise 1

A chemical reaction may yield either an alkaline or an acid result. The probability that the result is alkaline is p. A lab technician repeats the above reaction in independent conditions, and stops when both an alkaline and an acid result have occurred at least once. Let N_k and N_a be the number of alkaline and acid outcomes recorded by the technician at the end of her experiment.

1) Compute the CDF of N_k and N_a
2) Find the probability that N_k exceeds N_a, as a function of p. Draw a graph of the latter.
3) Compute the mean values of N_k and N_a. Check limit case and justify the answer.
4) State whether or not N_k and N_a are independent and justify your answer.

Exercise 2

A system serves job at a rate μ. When it is idle, it goes into power saving (immediately). On arrival of a new job, the system wakes up again. The wake-up operation takes an exponentially distributed time, with a mean $1/\beta$. During wake-up, the system does not accept jobs. Jobs arrive at a rate λ.

1) Model the system as a queueing system and draw its CTMC;
2) Compute the steady-state probabilities and the stability condition. State explicitly what happens of the stability condition when β increases;
3) Compute the mean number of jobs in the system;
4) Compute the mean response time of a job. Draw a graph of the latter with β on the abscissa.

Exercise 1 – Solution

1) Let us start from N_k.

$$P\{N_k = 1\} = \sum_{n=1}^{+\infty} P\{n \text{ consecutive acid and one alkaline}\} + P\{\text{one alkaline and one acid}\}$$

$$= \sum_{n=1}^{+\infty} (1-p)^n \cdot p + p \cdot (1-p) = (1-p) \cdot p \cdot \left(\frac{1}{p} + 1\right) = (1-p) \cdot (1+p) = 1 - p^2$$

And, for $n > 1$, $P\{N_k = n\} = P\{n \text{ consecutive acid and one alkaline}\} = p^n \cdot (1-p)$

The CDF of N_k is:

$$F_k(1) = 1 - p^2$$

$$F_k(n) = 1 - p^2 + \sum_{i=2}^{n} p^i \cdot (1-p) = (1-p)\left[\frac{1-p^{n+1}}{1-p} - (1+p) + (1+p)\right] = 1 - p^{n+1}$$

Therefore, it is $F_k(n) = 1 - p^{n+1} \ \ \forall n \geq 1$

Define $q = 1 - p$: for N_a it is $P\{N_a = 1\} = (1-q) \cdot (1+q) = p \cdot (2-p)$, $P\{N_a = n\} = q^n \cdot p = (1-p)^n \cdot p$.

Symmetrically, it is $F_a(n) = 1 - (1-p)^{n+1} \ \ \forall n \geq 1$

2) Event $\{N_k > N_a\}$ occurs when three or more reactions are observed, and the last one is acid. Hence:

$$P\{N_k > N_a\} = \sum_{n=2}^{+\infty} P\{N_k = n\} = \sum_{n=2}^{+\infty} p^n \cdot q = 1 - q \cdot (1+p) = p^2$$

3) From the formula, we obtain:

$$E[N_k] = 1 \cdot P\{N_k = 1\} + \left[\sum_{n=2}^{+\infty} n \cdot P\{N_k = n\}\right] =$$
$$(1-p) \cdot (1+p) + (1-p) \cdot \left[\left(\sum_{n=2}^{+\infty} n \cdot p^n\right)\right] =$$
$$(1-p) \cdot (1+p) + (1-p) \cdot \left[\left(\sum_{n=1}^{+\infty} n \cdot p^n\right) - p\right] =$$
$$(1-p) \cdot \left[\left(\frac{p}{(1-p)^2} - p + 1 + p\right)\right] =$$
$$\frac{p}{1-p} + (1-p)$$

We have $lim_{p\to 1} E\,[N_k] = +\infty$. This can be explained by observing that, if no acid outcome ever appears, the sequence of alkaline ones becomes infinitely long. Furthermore, we have $lim_{p\to 0} E\,[N_k] = 1$. In fact, if no alkaline outcome ever appears, the only sequences that we may ever obtain are infinitely long sequences of acid outcomes, *terminated by an alkaline one anyway* (this is mandatory for the experiment to terminate). Hence the mean number of alkaline outcomes will be equal to one.

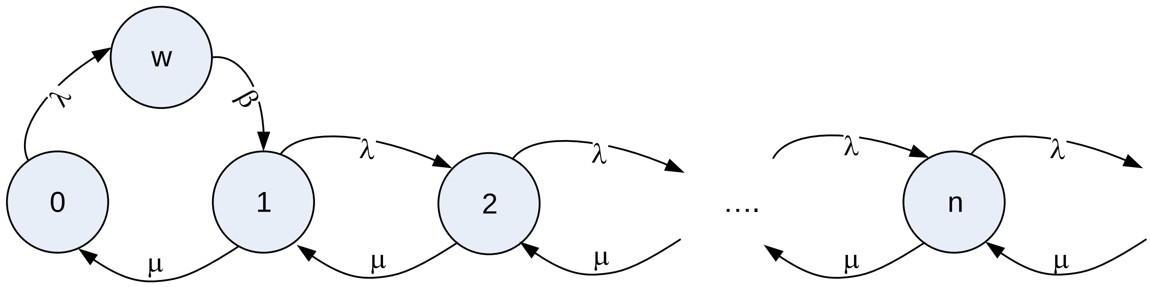
Symmetrically, we get:

$$E[N_a] = \frac{1-p}{p} + p$$

And symmetric explanations hold in limit cases, mutatis mutandis.

4) The two RVs are *not* independent. In fact, $P\{N_k = a, N_a = b\} = 0$ if $a > 1, b > 1$. On the other hand, it is $P\{N_k = a\} \cdot P\{N_a = b\} \neq 0$.

Exercise 2 - Solution



State w is the one during which the system wakes up. Global equilibrium equations in states 0 and w yield the following:

$$p_0 \cdot \lambda = p_1 \cdot \mu,\ p_w \cdot \beta = p_0 \cdot \lambda$$

And then we have

$$p_j \cdot \lambda = p_{j+1} \cdot \mu,\ \ j \geq 1$$

After a few straightforward manipulations, normalization reads:

$$p_0 \cdot \left[1 + \frac{\lambda}{\beta} + \sum_{j=1}^{+\infty} \left(\frac{\lambda}{\mu}\right)^j\right] = 1$$

Call $u = \lambda/\mu$ for simplicity. The stability condition is $u < 1$, and does not depend on β. SS probabilities are:

$$p_0 = \frac{(1-u) \cdot \beta}{\beta + \lambda \cdot (1-u)}$$

$$p_w = \frac{(1-u) \cdot \lambda}{\beta + \lambda \cdot (1-u)}$$

$$p_j = \frac{(1-u) \cdot \beta \cdot u^j}{\beta + \lambda \cdot (1-u)},\ \ j \geq 1$$

To compute the mean number of jobs in the system, one has to remember that the system contains one job in state w as well. Therefore:

$$E[N] = \sum_{j=1}^{+\infty} j \cdot p_j + 1 \cdot p_w = \cdots = 1 + \frac{2u-1}{1-u} \cdot \frac{1}{1 + \frac{\lambda}{\beta}(1-u)}$$

One can check that $\lim_{\beta \to +\infty} E[N] = \frac{u}{1-u}$, and that $\lim_{\beta \to 0} E[N] = 1$. Both limits make sense intuitively.

In order to compute the mean response time, one needs to remember that this is not a PASTA system, since the system does not accept jobs when in state w. Therefore, it is $\bar{\lambda} = \lambda \cdot (1 - p_w) = \frac{\lambda}{1+\frac{\lambda}{\beta}(1-u)}$. This said, one can apply Little's law and find:

$$E[R] = \frac{E[N]}{\bar{\lambda}} = \cdots = \frac{1}{\lambda} \cdot \frac{u}{1-u} + \frac{1-u}{\beta}$$

The first term in the above expression is the mean response time of a standard M/M/1. The second term is due to the added wake-up time, and goes to zero as β increases.