Exercise 1

Consider the following function:

$$F(x) = \begin{cases} 0 & x \leq -5 \\ \dfrac{\alpha \cdot x + 5}{\beta + |x|} & x > -5 \end{cases}$$

Where α, β are *positive* constants.

1) Determine under what conditions $F(x)$ is a CDF.

Assume from now on that we are in the above conditions.

2) Compute the PDF of RV X, whose CDF is $F(x)$.
3) Determine under what further conditions $E[X]$ is finite.
4) Assuming $\beta = 5$, compute $E[X]$.

Exercise 2

A service is hosted on a system having K identical servers. A job dispatcher routes each arriving job to an idle server, if there exists one, and *rejects it* otherwise. Assume that the service time and interarrival time are exponentially distributed RVs, whose rates are μ, λ respectively.

1) Draw the CTMC (or transition rate diagram).
2) Compute the stability condition and an expression for the steady-state probabilities.
3) Find under what conditions the PMF of the steady-state probability is:
   a. strictly increasing , i.e., $p_{j+1} > p_j$, $0 \leq j < K$.
   b. strictly decreasing.
   c. Neither of the above.

   Explain your findings.

4) Compute the loss probability
5) Compute the mean number of busy servers and, from that, the mean response time. Discuss both expressions when $K \to \infty$.
6) Compute the steady-state probabilities and the performance indexes for $K = 1$. Discuss the result.

Exercise 1 – Solution

1) In order to be a CDF, the following should happen:
   a) $F(x)$ must be monotonic
   b) $lim_{x\to-\infty} F(x) = 0$
   c) $lim_{x\to+\infty} F(x) = 1$

b) always holds. c) holds if and only if $\alpha = 1$.
As far as monotonicity is concerned, we observe that $F(x)$ is identically null for $x \leq -5$, and that its derivative is:

$$F'(x) = \begin{cases} \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

when $x > -5$. Therefore, monotonicity is guaranteed if $\beta \geq 5$.
The conditions requested by 1) are $\alpha = 1$, $\beta \geq 5$.

2) As per the computations above, it is:

$$f(x) = \begin{cases} 0 & x < -5 \\ \dfrac{\beta+5}{(\beta-x)^2} & -5 < x < 0 \\ \dfrac{\beta-5}{(\beta+x)^2} & x > 0 \end{cases}$$

3) It is:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \int_{-5}^{0} x \cdot \frac{\beta+5}{(\beta-x)^2}dx + \int_{0}^{+\infty} x \cdot \frac{\beta-5}{(\beta+x)^2}dx =$$
$$(\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx$$

Now, the first integral is always finite (since its limits are), whereas the second may not be. After few algebraic passages, we obtain:

$$\int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx = \int_{0}^{+\infty} \frac{(\beta+x)-\beta}{(\beta+x)^2}dx = \left[\frac{\beta}{\beta+x} + log|\beta+x|\right]_{0}^{+\infty} = +\infty$$

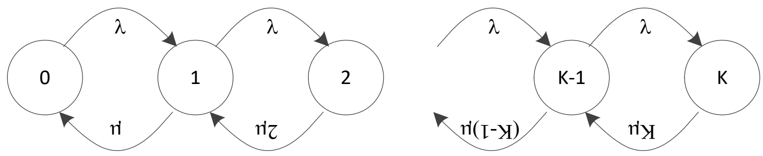
Therefore, the expectation exists only if $\beta = 5$.

4) Assuming $\beta = 5$, the expectation is equal to:

$$E[X] = (\beta+5) \cdot \int_{-5}^{0} \frac{x}{(\beta-x)^2}dx + (\beta-5) \cdot \int_{0}^{+\infty} \frac{x}{(\beta+x)^2}dx$$
$$= 10 \cdot \int_{-5}^{0} \frac{x}{(5-x)^2}dx = 10 \cdot \left[\frac{-5}{x-5} + log|x-5|\right]_{-5}^{0}$$
$$= 5 - 10\, log(2)$$

Exercise 2 – Solution

1) The diagram is shown below



2) The system is finite, hence always stable. The steady-state probabilities can be computed by writing the local balance equations: $\lambda \cdot p_j = (j+1)\cdot \mu \cdot p_{(j+1)}$, $0 \le j \le K-1$. From the above, we quickly obtain:

$$p_j = \frac{1}{j!}\cdot\left(\frac{\lambda}{\mu}\right)^j \cdot p_0,\ 0 \le j \le K.$$

Call $u = \lambda/\mu$. By imposing the normalization condition, we get:

$$p_0 = \frac{1}{\sum_{i=0}^{K}\frac{u^i}{i!}},\ p_j = \frac{\frac{u^j}{j!}}{\sum_{i=0}^{K}\frac{u^i}{i!}}.$$

3) $p_{j+1} > p_j \Leftrightarrow \frac{\frac{u^{j+1}}{(j+1)!}}{\sum_{i=0}^{K}\frac{u^i}{i!}} > \frac{\frac{u^j}{(j)!}}{\sum_{i=0}^{K}\frac{u^i}{i!}} \Leftrightarrow u > j+1$. Since the above must hold for every j up to $K-1$ included, the required condition is $u > K$. This can be explained by observing that, under this condition, the arrival rate is such that the serving capacity of the system (which is $K\mu$ under a heavy load) is smaller than the arrival rate.

Using the same reasoning, we find that the PMF is decreasing when $u < 1$. This means that a single server has enough capacity to serve the arrivals, hence it is increasingly more unlikely that more servers will be busy. When $1 \le u \le K$, the PMF is neither (strictly) increasing nor (strictly) decreasing.

4) The loss probability is $p_K = \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K}\frac{u^i}{i!}}$.

5) The mean number of busy servers is:

$$E[N] = \sum_{j=1}^{K} j\cdot p_j = \frac{\sum_{j=1}^{K} j\frac{u^j}{j!}}{\sum_{i=0}^{K}\frac{u^i}{i!}} = \frac{u\cdot\sum_{j=1}^{K}\frac{u^{j-1}}{(j-1)!}}{\sum_{i=0}^{K}\frac{u^i}{i!}} = u\cdot\frac{\left[\sum_{j=0}^{K}\frac{u^j}{j!} - \frac{u^K}{K!}\right]}{\sum_{i=0}^{K}\frac{u^i}{i!}} = u\cdot(1-p_K)$$

The mean response time can be obtained through Little's theorem, using the *mean* arrival rate $\bar{\lambda} = \lambda \cdot (1 - p_K)$. Thus, the average response time is $E[R] = E[N]/\bar{\lambda} = \frac{u \cdot (1 - p_K)}{\lambda \cdot (1 - p_K)} = \frac{1}{\mu}$. In fact, there is never any queueing in this system.

When $K \to \infty$ the system becomes an $M/M/\infty$. For this system, the mean number of busy servers is in fact $u = \lambda/\mu$. This can only happen if $\lim_{K\to\infty} p_k = 0$. Note that this condition is necessary of *any* PMF with an infinite support, hence must hold for this PMF as well. We can check it explicitly:

$$\lim_{K\to\infty} p_K = \lim_{K\to\infty} \frac{\frac{u^K}{K!}}{\sum_{i=0}^{K} \frac{u^i}{i!}} = e^{-u} \cdot \lim_{K\to\infty} \frac{u^K}{K!} = 0$$

6) When $K = 1$ we have $p_0 = \frac{1}{\sum_{i=0}^{K} \frac{u^i}{i!}} = \frac{1}{1+u}$, $p_1 = \frac{\frac{u^1}{1!}}{1+u} = \frac{u}{1+u}$. It is $p_K = p_1 = \frac{u}{1+u}$, $E[N] = p_1 = \frac{u}{1+u}$, and also $E[N] = u \cdot (1 - p_K) = u \cdot \left(1 - \frac{u}{1+u}\right) = u \cdot \left(\frac{1}{1+u}\right) = \frac{u}{1+u}$, $E[R] = \frac{1}{\mu}$. In this case, the system is an $M/M/1/1$ (or a 1-buffer), and the reader can easily check that the above expressions match those of such a system.