PECSN, 16/6/20

Exercise 1

A hospital in a large city admits both CoVid19 and standard patients. The mean daily intake of patients of either type is λ and μ respectively. The numbers of patients of the two types are independent of each other.

1) Compute the probability that the hospital admits patients in a day.
2) Compute the probability that the total number of patients is m, and h of them are CoVid19 patients.
3) The hospital registry shows that the total number of patients admitted today is m. What is the probability that h of them are CoVid19 patients? Justify your result.
4) Assume that $\lambda = \mu = 20$. Compute the probability of having at least 60 patients of either type, given that you have 100 patients. Discuss the correctness of the approximation.

Exercise 2

Following CoVid19 restrictions, a restaurant operates by hosting at most K customers simultaneously. Customers can order a meal including one to three courses (*starter*, *main*, *dessert*). After eating, they pay at the counter and leave. As soon as a customer leaves, a new one is admitted in (there is always a queue outside the restaurant).

Half of the customers choose to begin their meal with a starter; one fourth of the customers forgo the starter and order directly the main, and the remaining fourth order only the dessert. After eating a starter or a main, half of the customers choose to follow on with the next course, whereas the other half check out.

Assume that the restaurant has three cooks, each specialized in one course, which process their orders sequentially. Each cook takes an exponentially distributed time to serve one order, and let its mean be $1/\mu$. Assume that the check-out operation is twice as fast.

1) Model the system as a queueing network;
2) Solve the routing equations and compute the SS probabilities in their general form;
3) Compute the normalizing constant as a function of the number of customers K;
4) Compute the utilization of the cooks and the cashier; check the formula in limit cases.
5) Compute the number of customers served per unit of time;

Exercise 1 - Solution

1) It stands to reason that the number of patients of either type is distributed as a Poisson RV - large number of “trials” (citizens), low “success probability” (i.e., getting ill). The sum of independent Poisson variables is a Poisson variable, whose mean is $E[Z] = \lambda + \mu$. Therefore, $P\{Z > 0\} = 1 - P\{Z = 0\} = 1 - e^{-(\lambda+\mu)}$

2) The probability is the following:

$$\begin{aligned} P\{X = h, Z = m\} &= P\{X = h, Y = m - h\} \\ &= P\{X = h\} \cdot P\{Y = m - h\} \\ &= \left(e^{-\lambda} \cdot \frac{\lambda^h}{h!}\right) \cdot \left(e^{-\mu} \cdot \frac{\mu^{m-h}}{(m-h)!}\right) \\ &= e^{-(\lambda+\mu)} \cdot \frac{\lambda^h \mu^{m-h}}{h!\,(m-h)!} \end{aligned}$$

3) Using the former result, we get:

$$P\{X = h | Z = m\} = \frac{P\{X = h, Y = m - h\}}{P\{Z = m\}} = \frac{e^{-(\lambda+\mu)} \cdot \frac{\lambda^h \mu^{m-h}}{h!\,(m-h)!}}{e^{-(\lambda+\mu)} \cdot \frac{(\lambda+\mu)^m}{m!}} = \binom{m}{h} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^h \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{m-h}$$

The above distribution is clearly a binomial, with $p = \frac{\lambda}{\lambda+\mu} < 1$. In fact, once you fix the total number of patients, the number of CoVid19 ones is the count of a repeated Bernoullian experiment, whose success rate is the probability that an admitted patient has CoVid19. This probability is in fact $\frac{\lambda}{\lambda+\mu}$.

4) The requested probability is:

$$\begin{aligned} &P\big\{\{X \geq 60 | Z = 100\} \cup \{X \leq 40 | Z = 100\}\big\} \\ &= 1 - P\{40 \leq X \leq 60 | Z = 100\} \end{aligned}$$

The above probability can be approximated via a Gaussian, as long as $m \cdot p \cdot (1-p) > 10$. In fact, with $m = 100$ and $p = \frac{\lambda}{\lambda+\mu} = 0.5$ we get $m \cdot p \cdot (1-p) = 25 > 10$, hence the approximation is sound. Thus, we have:

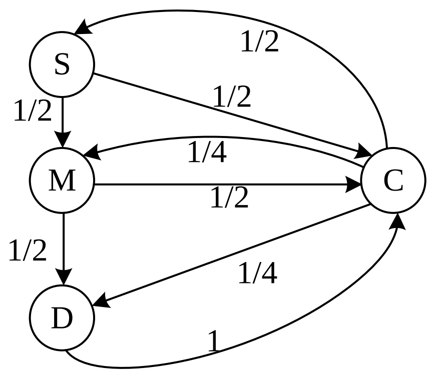
$$\begin{aligned} P\{40 \leq X \leq 60 | Z = 100\} &= P\left\{\frac{39.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{X - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} \leq \frac{60.5 - m \cdot p}{\sqrt{m \cdot p \cdot (1-p)}} | Z = 100\right\} \\ &= \Phi(2.1) - \Phi(-2.1) = -1 + 2 \cdot \Phi(2.1) \end{aligned}$$

The requested probability is thus $1 - [-1 + 2\Phi(2.1)] = 2\big(1 - \Phi(2.1)\big) \simeq 0.0357$.

Exercise 2 - Solution

The system can be modeled by a closed QN, with four SCs (Starters' cook, Mains' cook, Desserts' cook, Cashier –S, M, D, C for short from now on).

The QN diagram and routing matrix are the following:



$$\begin{array}{c c} & \begin{array}{cccc} S & M & D & C \end{array} \\ \begin{array}{c} S \\ M \\ D \\ C \end{array} & \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

The input/output balance relationships yield the following:

$$\lambda_S = \lambda_C/2$$
$$\lambda_M = \frac{\lambda_C}{4} + \frac{\lambda_S}{2} = \frac{\lambda_C}{2}$$
$$\lambda_D = \frac{\lambda_C}{4} + \frac{\lambda_M}{2} = \frac{\lambda_C}{2}$$

Therefore, a solution to the routing equations is $e^T = [e, e, e, 2e]^T$. Setting $e = \mu$, one finds $\rho^T = [1,1,1,1]^T$.

Therefore, we have $p(n_S, n_M, n_D, n_C) = \frac{1}{G(M,K)} \cdot 1^{n_S} \cdot 1^{n_M} \cdot 1^{n_D} \cdot 1^{n_C} = \frac{1}{G(M,K)}$

The normalizing constant is the cardinality of set $|\mathcal{E}|$, since all states are equally likely. Therefore, it is $G(M,K) = \binom{K+3}{3}$.

The utilization of the cooks and the cashiers is the same, and it is equal to $U = \frac{G(4,K-1)}{G(4,K)} = \frac{\binom{K+2}{3}}{\binom{K+3}{3}} = \frac{K}{K+3}$. The expression makes perfect sense, since the utilization grows to 1 with K.

The number of customers served per units of time is the throughput at the cashier. The latter is: $\gamma = 2\mu \cdot \frac{K}{K+3}$.