PECSN 25/02/2020

Name and surname: ____ *Matricola:* ________________

I will sit for the oral examination in: ☐ *February* ☐ *April (*)*

The above decision *<u>cannot</u>* *be changed later.*

(*) If you are ineligible, your written test will be discarded.

Exercise 1

Consider function:

$$f(x) = \begin{cases} -\alpha \cdot x^2 + 1 & x \in \left[-\frac{1}{\sqrt{\alpha}}; +\frac{1}{\sqrt{\alpha}}\right] \\ 0 & otherwise \end{cases}$$

1) Find α such that the above one is a PDF.
2) Let X be a RV having $f(x)$ as a PDF. Compute its mean, median and variance.
3) Let $Y = \log(X + \frac{1}{\sqrt{\alpha}})$. Compute $f_Y(y)$.

Exercise 2

A smart-service industry operates a fleet of N identical *robots*, each one composed of m electro-mechanic subsystems. When a robot is switched on, the mean-time-to-failure of subsystem j is exponential, with a rate λ_j, $1 \leq j \leq m$. As soon as one subsystem fails, the robot must be switched off, removed from operation and repaired. Repairs occur in a maintenance bay, which has a FIFO queue, and they take an exponential time whose mean is $1/\mu$, regardless of which subsystem is being repaired. After the repairment, the robot is switched on and put back into operation.

1) Model the system and draw its CTMC (or *transition-rate diagram*)
2) Compute the probability that there are n operating robots at the steady state, and the stability condition under which a steady state is reached.
3) Compute the mean number of operating robots.
4) Compute the mean downtime for a robot following a failure of one of its subsystems.

Exercise 1 – Solution

1) One should compute α such that:

$$\int_{-\frac{1}{\sqrt{\alpha}}}^{+\frac{1}{\sqrt{\alpha}}}(-\alpha\cdot x^2+1)dx=[\frac{-\alpha\cdot x^3}{3}+x]_{-\frac{1}{\sqrt{\alpha}}}^{+\frac{1}{\sqrt{\alpha}}}=1$$

After a few straightforward manipulations, one gets $\alpha = 16/9$.

2)Function $f(x)$ is even, hence its mean and median are null. From the above, one obtains

$$Var(X)=E[X^2]=\int_{-\frac{3}{4}}^{+\frac{3}{4}}x^2\left(-\frac{16}{9}\cdot x^2+1\right)dx=\frac{3}{80}$$

3) $Y=\log(X+\frac{3}{4})$. Therefore, Y is defined in $[-\infty;\log(3/2)]$, and

$$F_Y(k)=P\{Y\le k\}=P\left\{\log\left(X+\frac{3}{4}\right)\le k\right\}=P\left\{X+\frac{3}{4}\le e^k\right\}=P\left\{X\le e^k-\frac{3}{4}\right\}=F_X(e^k-\frac{3}{4})$$

It is:

$$F_X(x)=\int_{-\frac{3}{4}}^{x}\left(-\frac{16}{9}\cdot y^2+1\right)dy=[\frac{-16}{27}y^3+y]_{-\frac{3}{4}}^{x}=\left[-\frac{16}{27}x^3+x\right]-\left[\frac{16}{27}\cdot\frac{27}{64}-\frac{3}{4}\right]=-\frac{16}{27}x^3+x+\frac{1}{2}$$

Therefore:

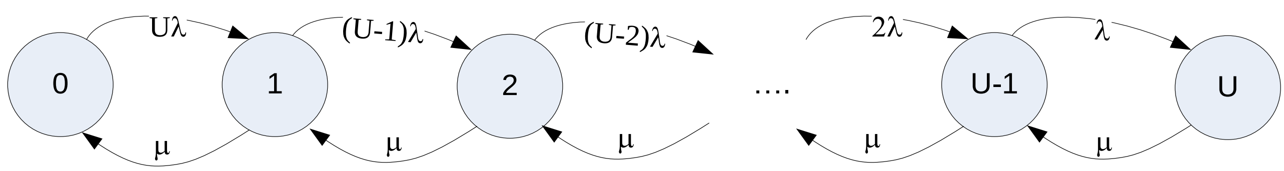
$$F_Y(k)=-\frac{16}{27}\left(e^k-\frac{3}{4}\right)^3+\left(e^k-\frac{3}{4}\right)+\frac{1}{2}=-\frac{16}{27}\left(e^{3k}-\frac{27}{64}-\frac{9}{4}e^{2k}+\frac{27}{16}e^k\right)+e^k-\frac{1}{4}$$
$$=-\frac{16}{27}e^{3k}+\frac{4}{3}e^{2k}$$

From the above, we get:

$$f_Y(k)=\frac{dF_Y(k)}{dk}=-\frac{16}{9}e^{3k}+\frac{8}{3}e^{2k}=\frac{8}{3}e^{2k}\left(1-\frac{2}{3}e^k\right)$$

Exercise 2 - Solution

1) The system is a finite population one, the population being N robots. Call $\lambda=\sum_{j=1}^{m}\lambda_j$. When all robots are operating, the arrival (i.e., subsystem failure) rate is thus $N\cdot\lambda$ due to independence. However, whenever one subsystem fails, the robot it belongs to is halted, hence there will be $N-1$ operating robots, hence an arrival rate of $(N-1)\cdot\lambda$, etc.
All in all, the CTMC can be deduced from the one of a finite-population system having N individuals (i.e., products) and an arrival rate λ. This is also consistent with the view that a robot can individually fail with a rate λ, since it contains independent subsystems whose failure rate are λ_j (recall the well-known theorem about the minimum of independent exponential distributions). We use the number of *repaired* robots as a state characterization (substitute N for U in the graph below):



2, 3,4) Once the above interpretation is understood, the computation for the above results can be found on my handouts (see "finite-population systems"). Here are the results:

- The system is always stable (since it has a finite population). The number of *operating* robots is the complement to N of the number of *repaired* robots.
  $p_{N-n} = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{N!}{(N-n)!} \cdot p_0$, $0 \leq n \leq N$ with $p_0 = \frac{1}{\sum_{j=0}^{N}\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{N!}{(N-j)!}}$, is the steady-state probability to have $N-n$ repaired robots, hence: $\pi_x = \frac{\left(\frac{\lambda}{\mu}\right)^{N-x} \cdot \frac{N!}{x!}}{\sum_{j=0}^{N}\left(\frac{\lambda}{\mu}\right)^j \cdot \frac{N!}{(N-j)!}}$, $0 \leq x \leq N$, is the SS probability to have *x* operating robots.
- The mean number of operating robots is $N - E[n] = \frac{\mu}{\lambda} \cdot (1 - p_0)$
- The mean downtime of a repaired robot is $E[R] = \frac{N}{\mu \cdot (1-p_0)} - \frac{1}{\lambda}$