Performance evaluation of Computer Systems and Networks – 15/01/2020

Exercise 1

The JPDF of a couple of continuous RVs X and Y is uniform in a triangle of the cartesian plane whose vertexes are $(-1;0), (+1;0), (0;+1)$. Define RV $Z = Y - X$.

1) State what values Z may assume, and compute its PDF;
2) Compute the mean value and the median of Z;
3) Compute the PDF of $W = 1/|Z|$. Is W a heavy-tailed RV? Justify your answer

Exercise 2

Consider a distributed system, where a central entity is connected to N independent computing agents. Each agent senses of a portion of space, and reports the sensed value to the central entity, which then elaborates an aggregate value by collating all the reports from the agents. The interaction between the components is as follows:

The **central entity**:

a) issues a "compute" request, which reaches all the agents simultaneously (in zero time);
b) waits until **all** the N sensed values have come in;
c) computes an aggregate value, taking an exponential time whose mean is $\frac{1}{\mu}$, and starts over.

Each of the $\boldsymbol{N}$ **agents**:

a) is initially idle, waiting for a "compute" request;
b) when it receives the "compute" request, it performs the sensing, which takes an exponential time whose mean is $\frac{1}{\lambda}$;
c) reports the sensed value to the central entity (in zero time), and then starts over.

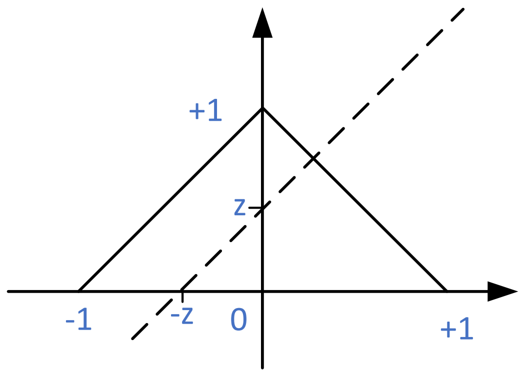
The candidate should:

1) Model the above system as a queueing system and draw its CTMC.
2) Compute the steady-state probabilities and the stability condition.
3) Compute the utilization of the central entity, and explain how it depends on N, λ, μ
4) Compute the probability that it takes less than α for all the N sensed values to come in, starting from a compute request.

Assume now that, instead of sending the "compute" request to all agents simultaneously, the central entity polls the agents *sequentially*, i.e., it schedules agent $j+1$ *right after* agent j has completed its sensing. Assume that polling takes zero time.

5) Draw the resulting CTMC and answer point 4) again, assuming as a reference instant the time when agent 1 is polled.

Exercise 1 – Solution



The JPDF is constant by definition, hence normalization on the triangle area implies that $f(x, y) = 1$.

RV Z is equal to value z when $Y - X = z$, i.e., for all the points on the dashed line in the figure. The values that Z may assume are therefore all the ordinates such that the dashed line intersects the triangle, i.e., $[-1; +1]$.

It is $P\{Z \le z\} = P\{Y - X \le z\}$. The latter is the probability that a point belongs to the triangle defined by the intersection of the original triangle with the dashed line, i.e., the one whose vertexes are $(-z; 0)$, $(+1; 0)$, $((1 - z)/2; (1 + z)/2)$. Since the JPDF is uniform and equal to one, this is just the area of the smaller triangle (normalized to the area of the larger triangle, which is however equal to one). That area is:

$$P\{Z \le z\} = P\{Y - X \le z\} = \frac{1}{2} \cdot \left[(1 + z) \cdot \frac{(1 + z)}{2}\right] = \frac{(1 + z)^2}{4}$$

The PDF of Z is obtained by differentiating the above:

$$f_Z(z) = \frac{1 + z}{2}$$

The mean value of Z is:

$$E[Z] = \int_{-1}^{+1} z \cdot \frac{1 + z}{2} dz = \left[\frac{z^2}{4} + \frac{z^3}{6}\right]_{-1}^{+1} = \frac{1}{3}$$

The median of Z is the solution to:

$$P\{Z \le z_{0.5}\} = \frac{1}{2}$$

Which is, after a few straightforward computations, $z_{0.5} = \sqrt{2} - 1$.

RV $|Z|$ may take on values in $[0; +1]$. Therefore, RV $W = 1/|Z|$ may take on values in $[1; +\infty]$, where we have $P\{W \le w\} = P\{1/|Z| \le w\} = P\{|Z| \ge 1/w\} = P\{Z \ge 1/w\} + P\{Z \le -1/w\}$.

Therefore, it is:

$$P\{W \le w\} = 1 - \frac{\left(1 + \frac{1}{w}\right)^2}{4} + \frac{\left(1 - \frac{1}{w}\right)^2}{4} = 1 - \frac{1}{w}$$

Therefore, it is $f_W(w) = \frac{1}{w^2}$. This is a heavy-tailed RV. In fact,

$$\forall \lambda > 0, \quad \lim_{x \to \infty} e^{\lambda x} \cdot \left(1 - F_W(x)\right) = \lim_{x \to \infty} \frac{e^{\lambda x}}{x} = \infty$$

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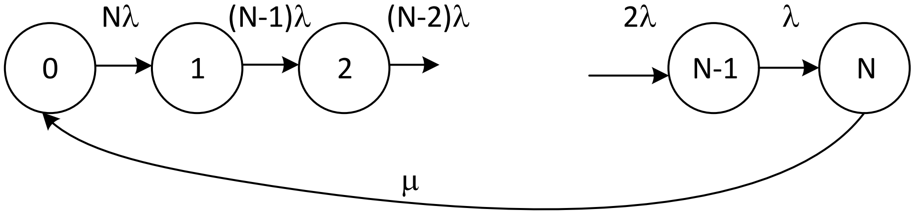
Exercise 2 – Solution

1) The system is a finite-population one with batch services. The CTMC is the following.



The system is always stable, since it has a finite number of states.

2) From the CTMC it is easy to write down equilibrium equations:

$$p_o \cdot N\lambda = p_N \cdot \mu$$
$$p_1 \cdot (N-1)\lambda = p_o \cdot N\lambda$$
$$\dots$$
$$p_j \cdot (N-j)\lambda = p_{j-1} \cdot (N-j+1)\lambda$$
$$\dots$$
$$p_N \cdot \mu = p_{N-1} \cdot \lambda$$

From which it is immediate to obtain:

$$p_j = p_0 \cdot \frac{N}{N-j} \quad (0 \leq j < N)$$
$$p_N = p_o \cdot N\frac{\lambda}{\mu}$$

We impose the normalization condition, and obtain:

$$p_0 \cdot N \cdot \left\{\frac{\lambda}{\mu} + \sum_{j=0}^{N-1} \frac{1}{N-j}\right\} = 1$$
$$p_0 \cdot N \cdot \left\{\frac{\lambda}{\mu} + \sum_{i=1}^{N} \frac{1}{i}\right\} = 1$$
$$p_0 = \frac{1}{N\left(H_N + \frac{\lambda}{\mu}\right)}$$

Where $H_N = \sum_{i=1}^{N} \frac{1}{i}$. From the above, we obtain:

$$p_j = \frac{1}{(N-j)\left(H_N + \frac{\lambda}{\mu}\right)} \quad (0 \leq j < N)$$
$$p_N = \frac{\frac{\lambda}{\mu}}{H_N + \frac{\lambda}{\mu}}$$

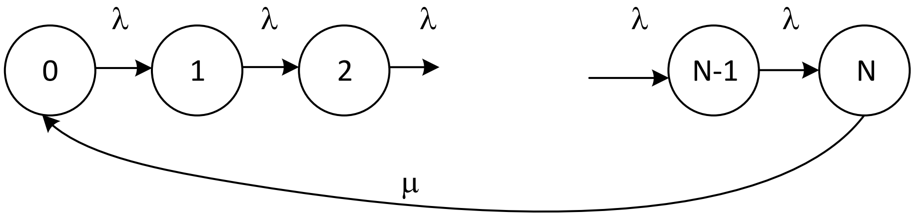
3) The central entity is idle except in state N (when it is computing the aggregate value). The answer is therefore:

$$p_{idle} = p_N = \frac{\frac{\lambda}{\mu}}{H_N + \frac{\lambda}{\mu}}$$

The utilization increases with λ, since when sensing is faster the central entity will spend comparatively more time doing its own computations. For the same reason, it decreases with μ. The utilization also decreases with N, since N influences the time before all the sensed values are in.

4) If X_i is the RV that models the sensing time at agent i, the time it takes for the central entity to obtain all the responses is $Z = \max_i\{X_i\}$. Since all agents are IID, It is $P\{Z \le \alpha\} = (P\{X_i \le \alpha\})^N$, i.e. $F_Z(\alpha) = \left(1 - e^{-\lambda\alpha}\right)^N$

5) In this case the CTMC is the following:



The time it takes to get all the sensed values in is the *sum* of N IID exponentials, which is an N-stage Erlang. Therefore,

$$F_Z(\alpha) = 1 - \sum_{k=0}^{N-1} e^{-\lambda\alpha} \frac{(\lambda\alpha)^k}{k!}$$