Effect of matching errors on the precision of ratios in integrated circuits.

Premise:

Let us define the ratio:

$$r = \frac{R_1 + ... + R_M}{R_{M+1} + ... + R_{M+N}} \tag{1}$$

Where R is a quantity (e.g. the resistance) associated to a type of component (e.g. a resistor) and R_i are the value assumed by R on $N+M$ nominally identical components. Clearly, the nominal value of r is M/N.
Let us now consider a series of different runs for the same chip, and identify the runs with number j. With obvious notation, we will indicate the ratio r_j at run j as:

$$r_j = \frac{R_{1,j} + ... + R_{M,j}}{R_{M+1,j} + ... + R_{M+N,j}} \tag{2}$$

For each run we can consider that the vaules $R_{i,j}$ are the result of a stochastic process. We can define the mean $<R_j>$ of that process. $<R_j>$ can be regarded as the average of an infinite number of nominally identical components $R_{i,j}$, all produced in the same chip at run j. This is clearly an ideal operation, since the number of component that can be included into a single chip in a single run is finite and generally small. We also observe that $<R_j>$ varies widely from run to run owing to process variations (global variations or <u>global errors</u>). We can write the $R_{i,j}$ values as:

$$R_{i,j} = < R_j > + \delta R_{i,j} \tag{3}$$

It can be easily demonstrated that, if the mean $<R_j>$ is really the limit of the average calculated on an ideally infinite number of <u>components</u>, the deviations $\delta R_{i,j}$ (local variations or <u>local errors</u>) are uncorrelated, i.e. the average $<\delta R_{i,j} \cdot \delta R_{k,j}>$ for $i \neq k$ calculated over an infinite number of <u>runs</u> (j=1,2,...,∞) is zero.

Using (3) we can write the ratio (2) as:

$$r_j = \frac{\left(< Rj > + \delta R_{1,j}\right) + ... + \left(< Rj > + \delta R_{M,j}\right)}{\left(< Rj > + \delta R_{M+1,j}\right) + ... + \left(< Rj > + \delta R_{M+N,j}\right)} = \frac{M < Rj > + \sum_{i=1}^{M} \delta R_{i,j}}{N < Rj > + \sum_{i=M+1}^{M+N} \delta R_{i,j}} \tag{4}$$

which can be rewritten as:

$$r_j = \frac{M}{N} \frac{1 + \sum_{i=1}^{M} \frac{\delta R_{i,j}}{< R_j >}}{1 + \sum_{i=M+1}^{M+N} \frac{\delta R_{i,j}}{< R_j >}} \qquad (5)$$

If $$\left| \sum_{i=1}^{M} \frac{\delta R_{i,j}}{< R_j >} \right| << 1 \quad \text{and} \quad \left| \sum_{i=M+1}^{M+N} \frac{\delta R_{i,j}}{< R_j >} \right| << 1 \qquad (8)$$

we can apply first order Taylor approximation and easily find that:

$$r_j \cong \frac{M}{N} \left(1 + \frac{\delta R_{i,j}}{M < R_j >} - \sum_{i=M+1}^{M+N} \frac{\delta R_{i,j}}{N < R_j >} \right) \qquad (9)$$

Now, considering that $\delta R_{i,j}$ are uncorrelated, we find:

$$\frac{\sigma_r}{r_{nom}} \cong \sigma_{\frac{\delta R}{R}} \sqrt{\frac{1}{N} + \frac{1}{M}} \qquad (10)$$

where r_{nom} is the nominal value of ratio r, i.e. M/N.
Eq. (10) relates the standard deviation of the relative error on the ratio r to the standard deviation of the relative mismatch error of the quantity R with respect of the average over a single run, $<Rj>$. Since the data that is generally reported is the standard deviation of the relative *matching error* between two elements, ΔR, than we have to find a relationship between $\sigma_{\delta R}$ and $\sigma_{\Delta R}$.

Considering for simplicity two elements R_1 and R_2, we can write:

$$\Delta R = R_{1,j} - R_{2,j} = \delta R_{1,j} - \delta R_{2,j} \qquad (11)$$

Using the statistical independence of the terms $\delta R_{i,j}$, we get:

$$\sigma_{\Delta R}^2 = \sigma_{\delta R_1}^2 + \sigma_{\delta R_2}^2 = 2\sigma_{\delta R}^2 \;\Rightarrow\; \sigma_{\delta R} = \frac{1}{\sqrt{2}} \sigma_{\Delta R} \qquad (12)$$

With simple substitutions, we can rewrite (10) as:

$$\frac{\sigma_r}{r_{nom}} \cong \sigma_{\frac{\Delta R}{R}} \left[\frac{1}{\sqrt{2}} \sqrt{\frac{1}{N} + \frac{1}{M}} \right] \qquad (13)$$