

erality of Theorem 1 side-steps the approximation problem except for putting the given data in real rational 2-variable form.

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# Sensitivity Analysis and Models of Nonlinear Circuits

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**Abstract**—Sensitivity analysis of general nonlinear circuits is considered using the concepts of state space. It is shown that sensitivity functions may be obtained by calculating the responses of a dependent linear model, topologically equivalent to the original, with element values and driving functions determined by partial derivatives of the characteristics and responses of the original circuit components. The sensitivity parameter may be taken as explicit in any one of the circuit parameters. The solution of the sensitivity functions from the model involves basically the same program as is used to obtain the solution of the original circuit and both solutions may be generated simultaneously.

#### INTRODUCTION

THE USE of auxiliary networks or sensitivity models for the sensitivity analysis of linear circuits and systems has been discussed by several authors. These models, as implemented on an analog or digital computer, dynamically generate sensitivity functions; that is, waveforms which represent the variation in the circuit or system response to be expected if an incremental change in a parameter were made. If the response is  $y(t, \alpha)$ , where  $\alpha$  is a circuit parameter, the sensitivity function, as used here, is given by  $\partial y(t, \alpha)/\partial \alpha$ . Kakotovic and Parezanovic, as discussed in a monograph by Tomovic [1], have introduced sensitivity models for systems represented by linear differential equations simulated on analog computers. Leeds [2] has expanded this concept

to linear-circuit analysis that may be carried out on a digital computer. This result may be shown to be essentially an application of the compensation theorem, as presented in a text by Skilling [4]. Recently, Leeds and Urgan [5], and Hachtel and Rohrer [6], have considered sensitivity functions applied to equivalent networks and circuit design, respectively. The extension of sensitivity studies to nonlinear circuits and systems has been discussed in papers by Meissinger [7], Dorf [8], and Rohrer [3], but without the physical interpretation of the sensitivity model and its relationship to the original circuit for computational purposes as presented here. The results obtained are summarized in the following theorem.

#### THEOREM

Sensitivity functions for a nonlinear circuit may be obtained by calculating the corresponding responses of a dependent circuit, topologically identical to the original, in which each component is replaced by a dependent linear equivalent given, at any instant, by the slopes of the voltage (current) versus current (voltage) characteristic for resistive (conductive) elements, the charge versus voltage characteristic for capacitive elements, and the flux versus current characteristic for inductive elements. The driving function for the sensitivity model depends upon the sensitivity parameter and is a voltage source in series with the component when the sensitivity parameter is explicit in resistive or inductive elements, a current source in parallel with the component when the sensitivity parameter is explicit in conductive or capacitive elements. It is always directed to cause a current flow in the

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sensitivity model element opposite to the direction of current through the element in the original circuit. The value of the source function is determined by the partial derivative of element voltage (current) with respect to the sensitivity parameter for resistive (conductive) elements; and by the time rate of change of the partial derivatives of charge with respect to the sensitivity parameter for capacitive elements and by flux with respect to the sensitivity parameter for inductive elements.

The results are generalized using state-space formulation, and it is shown that circuit responses and the sensitivity functions may be calculated simultaneously.

#### THEOREM DEVELOPMENT

Consider the following general matrix characterization for a circuit, as presented in a basic paper by Kuh and Rohrer [10].

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -0 & -F \\ F' & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} e \\ j \end{bmatrix} \quad (1)$$

$v_1$  and  $i_1$  represent link voltages and currents, respectively, and  $v_2$  and  $i_2$  represent tree-branch voltages and currents, respectively.  $F = [1 \mid B]$ , where  $B$  is the fundamental circuit matrix, and  $e$  and  $j$  represent independent voltage and current sources appearing in each fundamental loop and across each fundamental cutset, respectively. Following Kuh and Rohrer, if a normal tree is chosen, (1) may be written as follows:

$$\begin{bmatrix} v_S \\ v_R \\ v_L \\ i_C \\ i_G \\ i_\Gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -F_{SC} & 0 & 0 \\ 0 & 0 & 0 & -F_{RC} & -F_{RG} & 0 \\ 0 & 0 & 0 & -F_{LC} & -F_{LG} & -F_{L\Gamma} \\ F'_{SC} & F'_{RC} & F'_{LC} & 0 & 0 & 0 \\ 0 & F'_{RC} & F'_{LG} & 0 & 0 & 0 \\ 0 & 0 & F'_{L\Gamma} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_S \\ i_R \\ i_L \\ v_C \\ v_G \\ v_\Gamma \end{bmatrix} + \begin{bmatrix} e_S \\ e_R \\ e_L \\ j_C \\ j_G \\ j_\Gamma \end{bmatrix} \quad (2)$$

In (2), Kirchhoff's voltage law is written for each fundamental loop defined by links  $L$  (inductance),  $R$  (resistance), and  $S$  (inverse capacitance). Kirchhoff's current law is written for each fundamental cutset defined by tree branches  $\Gamma$  (inverse inductance),  $G$  (conductance), and  $C$  (capacitance). All voltage sources appear in voltage loop equations and all current sources in cutset equations.  $F$  (with appropriate subscripts) relates link and tree branch voltages and link and tree branch currents according to the network topology. A prime designates the inverse of a matrix.  $e_L$ ,  $e_R$ , and  $e_S$  represent independent voltage sources.  $j_\Gamma$ ,  $j_G$ , and  $j_C$  represent independent current sources.

The relationship between individual branch voltages and currents depends upon the nature of the component comprising that branch. For linear components these relationships may be written as

$$\begin{aligned} 1) \quad i_S &= \frac{d}{dt} (C_1 v_S) & 4) \quad i_G &= G_2 v_G \\ 2) \quad i_C &= \frac{d}{dt} (C_2 v_C) & 5) \quad v_L &= \frac{d}{dt} (L_{11} i_L + L_{12} i_\Gamma) \\ 3) \quad v_R &= R_1 i_R & 6) \quad v_\Gamma &= \frac{d}{dt} (L_{21} i_L + L_{22} i_\Gamma) \end{aligned} \quad (3)$$

where mutual inductances are treated with one coil as a link and the other as a tree branch. Dependent voltage and current sources may also be characterized by the following hybrid equations

$$v_R = H_{11} i_R + H_{12} v_G \quad (4a)$$

$$i_G = H_{21} i_R + H_{22} v_G \quad (4b)$$

with the voltage source treated as a link and the current source treated as a tree branch, dependent upon other branch currents and voltages as indicated.

For general nonlinear components, the foregoing expressions may be written as

$$\begin{aligned} 1) \quad i_S &= \frac{d}{dt} Q_{C1}(v_S) & 4) \quad i_G &= i_{G2}(v_G) \\ 2) \quad i_C &= \frac{d}{dt} Q_{C2}(v_C) & 5) \quad v_L &= \frac{d}{dt} [\phi_{11}(i_L) + \phi_{12}(i_\Gamma)] \\ 3) \quad v_R &= v_{R1}(i_R) & 6) \quad v_\Gamma &= \frac{d}{dt} [\phi_{21}(i_L) + \phi_{22}(i_\Gamma)]. \end{aligned} \quad (5)$$

For dependent sources

$$v_R = h_{11}(i_R) + h_{12}(v_G) \quad (6a)$$

$$i_G = h_{21}(i_R) + h_{22}(v_G). \quad (6b)$$

If an incremental change in a parameter  $\alpha$  occurs anywhere in the network, the effects upon voltages and currents may be obtained by considering the derivative of (2) with respect to  $\alpha$ . It is apparent that each of the variables is replaced by its derivative with respect to  $\alpha$ , ( $v_R$  by  $\partial v_R / \partial \alpha$ ,  $i_S$  by  $\partial i_S / \partial \alpha$ ,  $e_S$  by  $\partial e_S / \partial \alpha$ ,  $\dots$ , etc.), and that these derivatives are interrelated by the same topological matrix as the original network. Thus a new network is characterized, topologically identical to the original, with voltages and currents representing corresponding sensitivity functions. The exact nature of the branch voltage-current relationships in this sensitivity

model network depends upon the components of the original network and the explicit location of the parameter  $\alpha$ .

Consider that  $\alpha$  is explicit in only one of the independent sources. For example,  $e_s = e_s(t, \alpha)$ . From (5) and (6) it follows typically, that

$$\frac{\partial v_R}{\partial \alpha} = \frac{\partial v_{R1}(i_R)}{\partial \alpha} = \left( \frac{\partial v_{R1}(i_R)}{\partial i_R} \right) \frac{\partial i_R}{\partial \alpha} \quad (7a)$$

$$\frac{\partial i_C}{\partial \alpha} = \frac{\partial}{\partial t} \left( \frac{\partial Q_{C2}(v_C)}{\partial \alpha} \right) = \frac{\partial}{\partial t} \left[ \frac{\partial Q_{C2}(v_C)}{\partial v_C} \left( \frac{\partial v_C}{\partial \alpha} \right) \right] \quad (7b)$$

$$\begin{aligned} \frac{\partial v_L}{\partial \alpha} &= \frac{\partial}{\partial t} \left( \frac{\partial \phi_{11}(i_L)}{\partial \alpha} + \frac{\partial \phi_{12}(i_L)}{\partial \alpha} \right) \\ &= \frac{\partial}{\partial t} \left[ \frac{\partial \phi_{11}(i_L)}{\partial i_L} \left( \frac{\partial i_L}{\partial \alpha} \right) + \frac{\partial \phi_{12}(i_L)}{\partial i_L} \left( \frac{\partial i_L}{\partial \alpha} \right) \right]. \end{aligned} \quad (7c)$$

Comparison of (7) with (3) reveals that for the sensitivity network each resistance, capacitance, and inductance of the original network has been replaced by a dependent linear resistance, capacitance, or inductance with a value equal to the instantaneous slope of the defining component characteristic evaluated at the value of current or voltage of the element in the original circuit. Thus

$$\begin{aligned} 1) \quad C_{1 \text{ eq}} &= \frac{\partial Q_{C1}(v_S)}{\partial v_S} & 5) \quad L_{11 \text{ eq}} &= \frac{\partial \phi_{11}(i_L)}{\partial i_L} \\ 2) \quad C_{2 \text{ eq}} &= \frac{\partial Q_{C2}(v_C)}{\partial v_C} & 6) \quad L_{21 \text{ eq}} &= \frac{\partial \phi_{21}(i_L)}{\partial i_L} \\ 3) \quad R_{1 \text{ eq}} &= \frac{\partial v_{R1}(i_R)}{\partial i_R} & 7) \quad L_{12 \text{ eq}} &= \frac{\partial \phi_{12}(i_L)}{\partial i_L} \\ 4) \quad G_{2 \text{ eq}} &= \frac{\partial i_{C2}(v_C)}{\partial v_C} & 8) \quad L_{22 \text{ eq}} &= \frac{\partial \phi_{22}(i_L)}{\partial i_L}. \end{aligned} \quad (8)$$

For linear components these values are constants, namely,  $R_1$ ,  $C_1$ ,  $L_{11}$ ,  $L_{12}$ ,  $C_2$ ,  $G_2$ ,  $L_{21}$ , and  $L_{22}$ , respectively. The derivatives of the independent sources in (2) are zero, except for the source dependent upon the parameter,  $\alpha$ . In the sensitivity network this source is replaced by an equivalent source given by  $\partial e_s(t, \alpha)/\partial \alpha$ .

Consider that the variable  $\alpha$  is explicit in one of the resistive components so that one may write, for example,

$$v_R = v_{R1}(i_R, \alpha). \quad (9)$$

Now

$$\frac{\partial v_R}{\partial \alpha} = \frac{\partial v_{R1}(i_R, \alpha)}{\partial i_R} \left( \frac{\partial i_R}{\partial \alpha} \right) + \frac{\partial v_{R1}(i_R, \alpha)}{\partial \alpha}. \quad (10)$$

Comparison of (10) with (3c) reveals that in the sensitivity model the resistive component of the original network has been replaced by a dependent linear resistance  $R_{1 \text{ eq}} = \partial v_{R1}(i_R, \alpha)/\partial i_R$  in series with a voltage source of value  $e_{R1 \text{ eq}} = \partial v_{R1}(i_R, \alpha)/\partial \alpha$ . See Fig. 1(a). For a linear resistance,  $v_R = R_1 i_R$ . If  $\alpha = R_1$ , then  $R_{1 \text{ eq}} = R_1$  and  $e_{R1 \text{ eq}} = i_R$ . For a nonlinear resistance, if  $v_R = K i_R^2$  and  $\alpha = K$ , then  $R_{1 \text{ eq}} = 2K i_R$  and  $e_{R1 \text{ eq}} = i_R^2$ . The direction

of the voltage source is to cause current flow in a direction opposite to the current through the component in the original circuit. Since the derivatives of the independent sources in (2) with respect to  $\alpha$  are zero, they are not present in the sensitivity model. The other elements in the sensitivity model are given by their dependent linear equivalents as listed in (8).

Similarly, when the parameter  $\alpha$  is explicit in one of the capacitance elements,

$$i_C = \frac{\partial}{\partial t} Q_{C2}(v_C, \alpha) \quad (11)$$

$$\frac{\partial i_C}{\partial \alpha} = \frac{\partial}{\partial t} \left[ \frac{\partial Q_{C2}(v_C, \alpha)}{\partial v_C} \left( \frac{\partial v_C}{\partial \alpha} \right) \right] + \frac{\partial}{\partial t} \left( \frac{\partial Q_{C2}(v_C, \alpha)}{\partial \alpha} \right).$$

Comparison of (11) with (3b) reveals that in the sensitivity network the capacitance component of the original network has been replaced by a dependent linear capacitance  $C_{2 \text{ eq}} = \partial Q_{C2}(v_C, \alpha)/\partial v_C$  in parallel with a current source of value  $j_{C2 \text{ eq}} = (\partial/\partial t)[\partial Q_{C2}(v_C, \alpha)/\partial \alpha]$ . See Fig. 1(c). For a linear capacitance,  $Q_{C2}(v_C, \alpha) = C_2 v_C$ . If  $\alpha = C_2$ , then  $C_{2 \text{ eq}} = C_2$  and  $j_{C2 \text{ eq}} = dv_C/dt$ . For a nonlinear capacitance, if  $Q_{C2}(v_C, \alpha) = K v_C^2$  and  $\alpha = K$ , then  $C_{2 \text{ eq}} = 2K v_C$  and  $j_{C2 \text{ eq}} = d(v_C^2)/dt$ . The current source is directed to oppose the direction of current flow through the component in the original circuit. The derivatives of the independent sources in (2) are zero and they do not appear in the sensitivity model. The other components are given by their dependent linear equivalents as listed in (8).

The foregoing analysis may be applied to each type of component of (5), including the dependent sources of (6). The results are summarized in Fig. 1. In view of the foregoing, the theorem follows.

#### COMPUTATIONAL ASPECTS

For computational purposes it is important to note that when nonlinear components are present the dependent linear equivalents in the sensitivity model are essentially time varying since the slope of the element's nonlinear characteristic varies with its voltage or current. However, in a digital computation, the slope is usually available at each instant of computation time in the solution of the original circuit. For example, in order to obtain a solution of a diode or transistor circuit, the resistance of the nonlinear element has usually been replaced by a linear resistance used in the convergence process at any instant of computation time to obtain the computer solution. Also, the source currents or voltages for the sensitivity model can be calculated directly from the solution of the original circuit as it is generated. The foregoing may be demonstrated by considering the state equations for a circuit written in the following form where nonlinear resistance elements have been separated as dependent sources [11]. All other circuit elements are considered to be linear.

$$\frac{\partial x(t)}{\partial t} = A(\alpha)x(t) + B_1 u(x, \beta, t) + B_2 u_0(t) \quad (12)$$

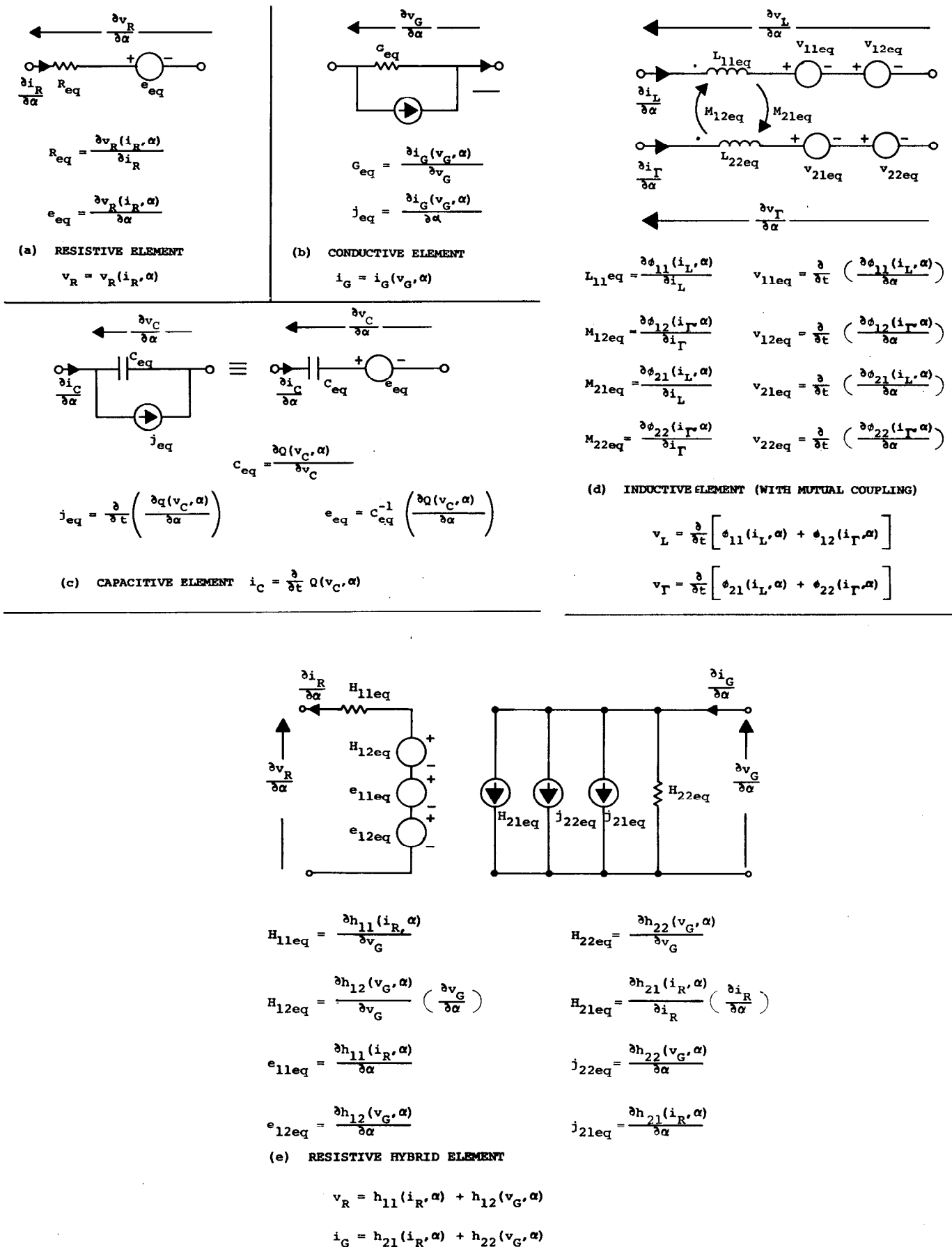


Fig. 1. Sensitivity model equivalents.

where  $x(t)$  is the state vector,  $A(\alpha)$  is the system matrix involving linear elements only,  $u(x, \beta, t)$  is a dependent source input vector,  $u_0(t)$  is an independent source vector, and  $B_1$  and  $B_2$  are input matrices.  $\alpha$  and  $\beta$  designate circuit sensitivity parameters. Considering variations with respect to  $\alpha$ , it follows from (12) that

$$\frac{\partial}{\partial t} \left( \frac{\partial x(t)}{\partial \alpha} \right) = \left( A(\alpha) + B_1 \frac{\partial u(x, \beta, t)}{\partial x} \right) \frac{\partial x(t)}{\partial \alpha} + \left( \frac{\partial A(\alpha)}{\partial \alpha} \right) x(t). \quad (13)$$

Equation (13) is the state equation for the sensitivity vector. The term  $B_1 \partial u(x, \beta, t)/\partial x$  represents the dependent resistance (conductance) of the nonlinear element and  $\partial A(\alpha)/\partial \alpha$  represents a coupling matrix that relates sensitivity model sources to the original circuit responses.

Considering variations with respect to  $\beta$ , it follows from (12) that

$$\frac{\partial}{\partial t} \left( \frac{\partial x(t)}{\partial \beta} \right) = \left( A(\alpha) + B_1 \frac{\partial u(x, \beta, t)}{\partial x} \right) \left( \frac{\partial x(t)}{\partial \beta} \right) + B_1 \frac{\partial u(x, \beta, t)}{\partial \beta}. \quad (14)$$

Equation (14) is the state equation for the sensitivity vector. The term  $B_1 \partial u(x, \beta, t)/\partial x$  represents the dynamic resistance (conductance) of the nonlinear element, and  $B_1 \partial u(x, \beta, t)/\partial \beta$  represents a coupling matrix that relates sensitivity model sources to the circuit responses. If (12) is integrated numerically, then at each instant of discrete time, the slope  $\partial u(x, \beta, t)/\partial x$  is known, and  $\partial u(x, \beta, t)/\partial \beta$  may be calculated. Using these values in (13) and (14)

enables the sensitivity functions to be integrated over each discrete interval of calculation. The solutions to (13) and (14) are dependent upon the solution to (12) and may be integrated simultaneously provided that the coefficient matrix and source vectors are reevaluated at each discrete interval as the numerical solution proceeds. When nonlinear terms are not present, the coefficient matrices are constant and only the source vectors of (13) and (14) need to be coupled to the solution of (12).

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