1 Practical procedures for dealing with noise spectral densities

1.1 Idealized amplifier noise and rms calculation.

Figures 1.1(a) and 1.1(b) show a typical noise spectrum in linear and logarithmic coordinates. The spectrum includes a $1/f$ (flicker noise) contribution at low frequency and a region of constant noise (broad-band noise, S_{XBB}), at higher frequency. The upper limit of the spectrum is indicated with B. For frequencies higher than B the spectral density decreases, tending to zero. For simplicity, in this course we will consider that the spectral density drops to zero abruptly for f>B. In real cases, the spectral density will gradually decrease, as shown by the dashed line in the figure.

In terms of total noise power, the approximation of abrupt upper band limit is valid when B coincides with the equivalent noise bandwidth. For a frequency response with a dominant pole (first order low pass response), the equivalent noise bandwidth is equal to $(\pi/2)f_{-3dB}$, where f_{-3dB}, is the cut-off frequency (frequency at which the response is 3dB below the value in the pass-band).

The flicker noise component can be written as:

$$S_{XF}(f) = \frac{k_F}{f^{\gamma}} \tag{1.1}$$

where k_F is a constant parameter and γ an exponent that in many practical cases is close to one. In the rest of this document we will consider γ=1. Thus we can write:

$$k_F = f \cdot S_{XF}(f) \tag{1.2}$$

Eqn. (1.2) allows determining the k_F value from an experimental or simulated noise spectrum. In particular, if the noise spectrum at 1 Hz is dominated only by the flicker component, then k_F coincides (only numerically) with the value assumed by the spectrum at 1 Hz.

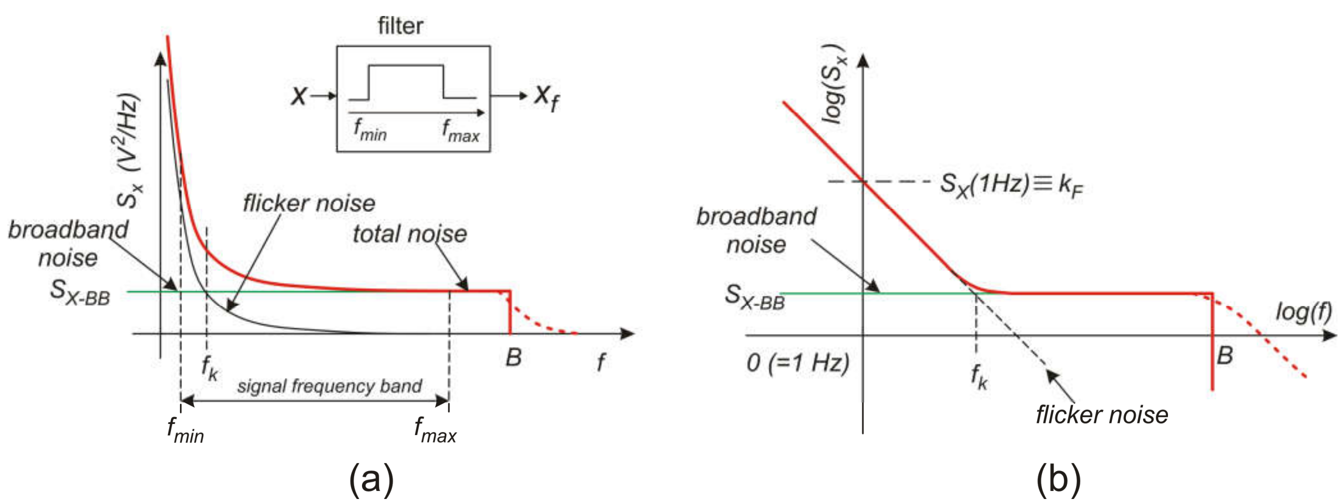

Fig. 1.1. Typical one-sided noise spectral density of an amplifier (a) and its representation in logarithmic scale. A possible signal bandwidth [f_{min}, f_{max}] is shown.

An important parameter of the noise spectrum shown in Fig.1.1 is the corner frequency, f_k, defined as the frequency at which the flicker component is equal to the broad-band (constant) component. Applying (1.2) at f_k:

$$f_k S_{XBB} = k_F \tag{1.3}$$

Let us consider that the readout channel has a bandwidth extending from f_{min} to f_{max}. In all well-designed acquisition systems, there is a filter, generally placed at the end of the processing chain, that limits the channel bandwidth to the minimum required by the signal. Any additional bandwidth with respect to this minimum is not only useless, but also harmful, since it increases the *rms* noise, degrading the system resolution. Therefore, we will suppose that the spectral noise superimposed to signal X is filtered by an ideal pass-band filter as sketched in the inset of figure 1.1(a). The target is calculating the *rms* noise at the output of the filter.

The *rms* noise in the signal bandwidth is given by:

$$x_{rms} = \sqrt{\int_{f_{\min}}^{f_{\max}} S_X(f)df} = \sqrt{\int_{f_{\min}}^{f_{\max}} S_{XBB}(f)df + \int_{f_{\min}}^{f_{\max}} S_{XF}(f)df} \tag{1.4}$$

The integral of the broadband component is simply given by:

$$\int_{f_{\min}}^{f_{\max}} S_{XBB}(f)df = S_{XBB}\left(f_{\max} - f_{\min}\right) \tag{1.5}$$

The integral of the flicker component is given by:

$$\int_{f_{\min}}^{f_{\max}} S_{XF}(f)df = k_F \ln\left(\frac{f_{\max}}{f_{\min}}\right) = k_F\, 2.3 \cdot n_{dec} \tag{1.6}$$

where n_{dec} is equal to:

$$n_{dec} == \log_{10}\left(\frac{f_{\max}}{f_{\min}}\right) = \text{number of decades between } f_{\min} \text{ and } f_{\max} \tag{1.7}$$

A problem with the flicker component could arise if the signal bandwidth extends down to DC, since f_{min} should be set to zero, making the flicker component diverge to infinity. This is a false problem, since a real DC component should be constant over an infinite time interval, and therefore it is only a theoretical abstraction. In practical cases, we consider a signal to be DC when it stays constant over the whole observation time interval, which is a finite interval. If we indicate the observation time with T_{obs}, then we can recognize frequencies as low as:

$$f_{\min} \approx \frac{1}{T_{obs}} \tag{1.8}$$

For example, if the observation time is 100 s, then the minimum significant frequency is roughly 0.01 Hz. If we do not know the observation time, we can arbitrarily assume that it is a few tens of second long (e.g. 100 s as in the example). An error in the determination of the observation time does not produce important errors in the estimated *rms* noise, due to the logarithmic dependence present in Eq. (1.6).

1.2 Inclusion of the offset into the noise power spectral density

Figure 1.2 shows a frequently used simplified representation of noise spectra. This is only a symbolic representation since, in real cases, flicker noise diverges to infinity, when *f* tends to zero; furthermore, the upper band limit (*B*) is not as abrupt as in the figure, but more progressive.

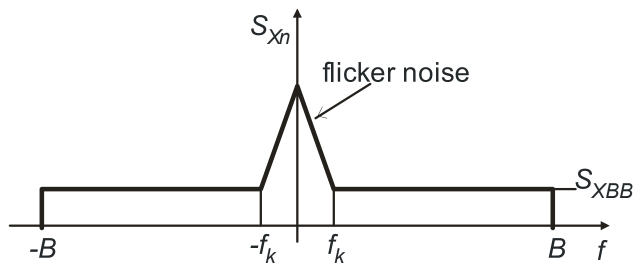

Fig. 1.2. Symbolic two-sided amplifier noise spectrum.

Nevertheless, this bilateral representation can be very useful to understand what happens to noise in modulation and sampling operations, where the spectra are replicated and shifted along the frequency axis. In order to extend this convenient representation to the other component of additive errors, i.e. offset, we can write the total error:

$$x_{ntot} = x_n + x_{io} \tag{1.9}$$

Note that even offset can be considered a stochastic process, consisting in a DC signals with random (DC) value. The autocorrelation function of the total additive error is given by:

$$R_{Xntot}(\tau) = < X_{ntot}(t) X_{ntot}(t-\tau) > = < (x_n(t) + x_{io})(x_n(t-\tau) + x_{io}) > \tag{1.10}$$

Thus:

$$R_{Xntot}(\tau) = < x_n(t) x_n(t-\tau) > + < x_{io}^2 > + < x_{io} x_n(t-\tau) > + < (x_n(t) x_{io}) > \tag{1.11}$$

Since offset and noise derives from different phenomena, they can be considered independent. Therefore we obtain the following result:

$$R_{Xntot}(\tau) = R_{Xn}(\tau) + \sigma_{Xio}^2 \tag{1.12}$$

where $R_{Xn}(\tau)$ is the noise correlation function, while σ_{Xio} is the offset standard deviation.

In terms of spectral density, the offset contribution becomes a Dirac delta function with value σ_{Xio}^2. Then we obtain the following symbolic representation that takes into account noise and offset together:

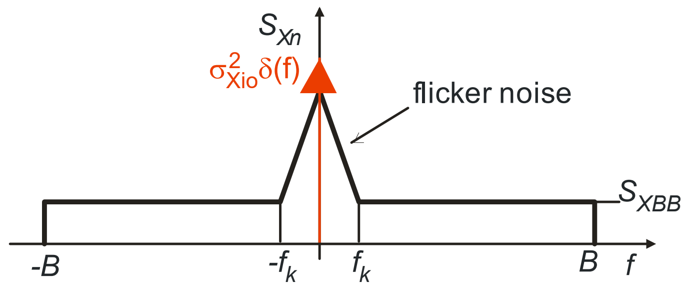

Fig 1.3. Offset and noise combined symbolic spectrum.