Effect of uncertainties on the angle estimation from the vector components.

Let us consider a vector ***V*** that lies on a plane (e.g.. the XY plane). With X and Y we will indicate the two components of vector ***V***. We will also suppose that X and Y are acquired with separate sensors and readout channels, and that we need to calculate the angle θ formed by the vector with the X-axis.
We can obtain θ from X and Y by means of the four-quadrant arctangent function, that produces angles between $-\pi$ and π:

$$\theta = \begin{cases} \arctan\left(\frac{Y}{X}\right) & \text{if } X > 0, Y > 0 \\ \arctan\left(\frac{Y}{X}\right) + \pi & \text{if } X < 0, Y > 0 \\ \arctan\left(\frac{Y}{X}\right) - \pi & \text{if } X < 0, Y < 0 \end{cases} \tag{1}$$

The four-quadrant arctangent (corresponding to the "atan2" function of the C math library) has to be used since the conventional arctangent function can produce only results between $-\pi/2$ and $\pi/2$ (two quadrants).
The error on the θ estimate is given by:

$$\theta_e = \left(\frac{\partial\theta}{\partial X} x_e + \frac{\partial\theta}{\partial Y} y_e\right) \tag{2}$$

Where x_e and y_e are the errors on the X and Y estimates.
Then:

$$\theta_e = -\frac{Y}{X^2}\frac{x_e}{1+\left(\frac{Y}{X}\right)^2} + \frac{1}{X}\frac{y_e}{1+\left(\frac{Y}{X}\right)^2} \tag{3}$$

This expression can be rewritten in the following way:

$$\theta_e = -\frac{Y \cdot x_e}{X^2+Y^2} + \frac{X \cdot y_e}{X^2+Y^2} \tag{4}$$

Note that the error on each axis is weighted by the value of the coordinate on the other axis. As a result, for example, the contribution of the noise from the X estimate will be maximum when X is zero. The same can be stated about channel Y. This is reasonable, if simple geometrical considerations are made.
In terms of standard deviations, if errors x_e and y_e are statistically independent we can write:

$$\sigma_{\theta e} = \frac{\sqrt{\sigma_{Ye}^2 X^2 + \sigma_{Xe}^2 Y^2}}{X^2+Y^2} \tag{5}$$

Where σ_{Xe} and σ_{Ye} and $\sigma_{\theta e}$ are the standard deviations of x_e,y_e and θ_e.
An interesting formula can be found when the errors on the estimates of X and Y are statistically equal, i.e. $\sigma_{Xe} = \sigma_{Ye} = \sigma_e$:

$$\sigma_{\theta e} = \frac{\sigma_e}{\sqrt{X^2+Y^2}} = \frac{\sigma_e}{\|\boldsymbol{V}\|} \tag{6}$$