Practical formulas for noise calculations

Figure 1(a) and (b) show a typical noise spectrum in linear and logarithmic coordinates. The spectrum includes a $1/f$ (flicker noise) contribution at low frequency and a region of constant noise (broad-band noise, S_{XBB}), at higher frequency. The upper limit of the spectrum is indicated with B. For frequencies higher than B the spectral density decreases, tending to zero. For simplicity, in this course we will consider that the spectral density drops to zero abruptly for f>B. In real cases, the spectral density will gradually decrease, as shown by the dashed line in the figure.

In terms of total noise power, the approximation of abrupt upper band limit is valid when B coincides with the equivalent noise bandwidth. For a frequency response with a dominant pole (first order low pass response), the equivalent noise bandwidth is equal to $(\pi/2)f_{-3dB}$, where f_{-3dB}, is the cut-off frequency (frequency at which the response is 3dB below the value in the pass-band).

The flicker noise component can be written as:

$$S_{XF}(f) = \frac{k_F}{f^{\gamma}} \tag{1}$$

where k_F is a constant parameter and γ an exponent that in many practical cases is close to one. In the rest of this document we will consider γ=1. Thus we can write:

$$k_F = f \cdot S_{XF}(f) \tag{2}$$

Eqn. (2) allows determining the k_F value from an experimental or simulated noise spectrum. In particular, if the noise spectrum at 1 Hz is dominated only by the flicker component, then k_F coincides (only numerically) with the value assumed by the spectrum at 1 Hz.

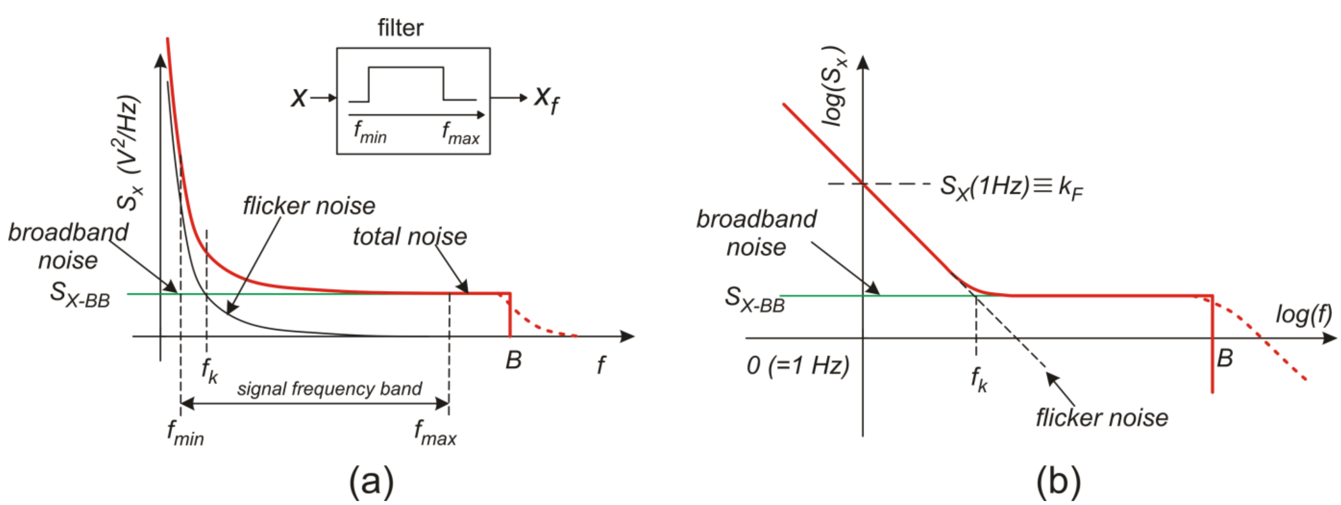

Fig. 1

An important parameter of the noise spectrum shown in Fig. 1 is the corner frequency, f_k, defined as the frequency at which the flicker component is equal to the broad-band (constant) component. Applying (2) at f_k:

$$f_k S_{XBB} = k_F \tag{3}$$

Let us consider that the readout channel has a bandwidth extending from f_{min} to f_{max}. In all well-designed acquisition systems, there is a filter, generally placed at the end of the processing chain, that limits the channel bandwidth to the minimum required by the signal. Any additional bandwidth with respect to this minimum is not only useless, but also harmful, since it increases the *rms* noise, degrading the system resolution. Therefore, we will suppose that the spectral noise superimposed to signal X is filtered by an ideal pass-band filter as sketched in the inset of figure 1(a). The target is calculating the *rms* noise at the output of the filter.

The *rms* noise in the signal bandwidth is given by:

$$x_{rms} = \sqrt{\int_{f_{min}}^{f_{max}} S_X(f)df} = \sqrt{\int_{f_{min}}^{f_{max}} S_{XBB}(f)df + \int_{f_{min}}^{f_{max}} S_{XF}(f)df} \tag{4}$$

The integral of the broadband component is simply given by:

$$\int_{f_{min}}^{f_{max}} S_{XBB}(f)df = S_{XBB}\left(f_{max} - f_{min}\right) \tag{5}$$

The integral of the flicker component is given by:

$$\int_{f_{min}}^{f_{max}} S_{XF}(f)df = k_F \ln\left(\frac{f_{max}}{f_{min}}\right) = k_F\, 2.3 \cdot n_{dec} \tag{6}$$

where n_{dec} is equal to:

$$n_{dec} = \log_{10}\left(\frac{f_{max}}{f_{min}}\right) = \text{number of decades between } f_{min} \text{ and } f_{max} \tag{7}$$

A problem with the flicker component could arise if the signal bandwidth extends down to DC, since f_{min} should be set to zero, making the flicker component diverge to infinity. This is a false problem, since a real DC component should be constant over an infinite time interval, and therefore it is only a theoretical abstraction. In practical cases, we consider a signal to be DC when it stays constant over the whole observation time interval, which is a finite interval. If we indicate the observation time with T_{obs}, then we can recognize frequencies as low as:

$$f_{min} \approx \frac{1}{T_{obs}} \tag{8}$$

For example, if the observation time is 100 s, then the minimum significant frequency is roughly 0.01 Hz. If we do not know the observation time, we can arbitrarily assume that it is a few tens of second long (e.g. 100 s as in the example). An error in the determination of the observation time does not produce important errors in the estimated *rms* noise, due to the logarithmic dependence present in Eq.(5).

The kT/C noise

This kind of noise occurs any time a voltage is sampled into a capacitor, as shown in Fig.2. When the switch S is closed, the voltage V_C is equal to the voltage of the source, indicated with V_0. For simplicity, we will consider that the voltage to be sampled is constant. When the switch opens (sampling instant t_c), the sampled voltage V_C is V_0+V_e, where V_ε is a random error.

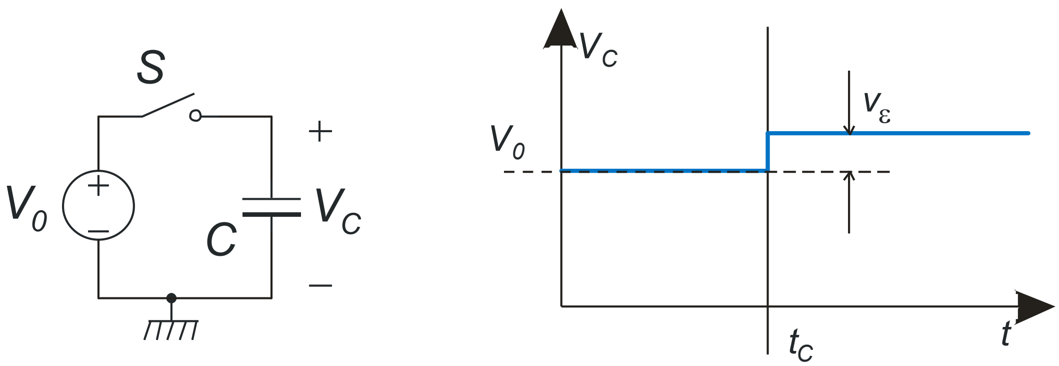

Fig.2

Repeating the sampling operation, as in Fig.3, where we have shown also the clock signal that controls the switch, we obtain different error voltages ($v_{\varepsilon 1}$, $v_{\varepsilon 2}$, $v_{\varepsilon 3}$ ...), demonstrating the random nature of the phenomenon.

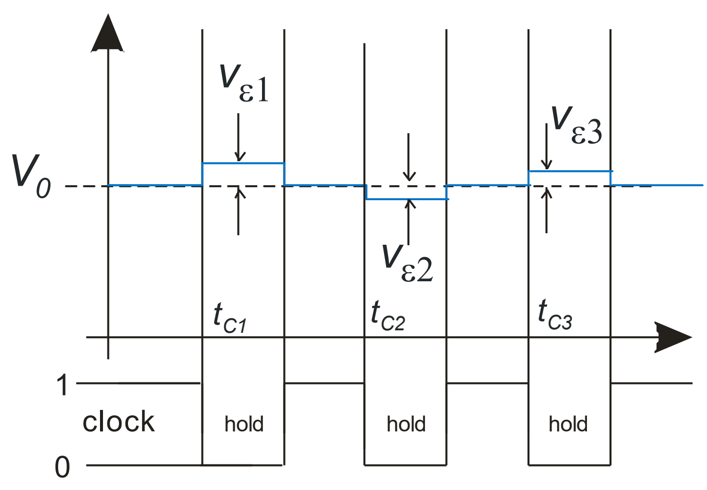

Fig. 3

This error is named "*kT/C*" ("*kT* over *C*") noise, since it is marked by the following property:

$$< v_{\varepsilon}^2 > = \frac{kT}{C} \tag{9}$$

where k is the Boltzmann constant and T the absolute temperature. The distribution of kT/C noise is gaussian.

The reason of the existence of the kT/C noise is very simple. Voltage V_0 is never really constant, but it is affected by thermal noise, associated to the series resistance of the voltage source. The real circuit, which includes also this resistance (R), is shown in Fig.4. When the switch is closed, the thermal noise voltage source of resistor R is filtered by capacitor C that, together with R, forms a first order low pass filter. The power spectral density (PSD) of the noise source is $4kTR$, so that the PSD of the noise voltage superimposed to V_C (v_{nc}) is given by:

$$S_{VC} = 4kTR \frac{1}{1+\left(\frac{f}{f_p}\right)^2} \qquad \text{with} \qquad f_p = \frac{1}{2\pi RC} \tag{10}$$

The mean square voltage of v_{nc} is given by the integral of the spectral density shown in (10) from 0 to infinity. By simple calculations:

$$< v_{nc}^2 > = \int_0^\infty S_{VC}(f)df = \frac{kT}{C} \tag{11}$$

Note that the mean square voltage of v_{nc} is independent of R. At the sampling instant, switch S opens and the voltage stored in C is the last value assumed by V_C, which includes also the v_{nc} noise contribution. Sampling of v_{nc} gives the sequence $v_{\varepsilon i}$ of Fig.3. This sequence has the same mean square value of v_{nc}, that is kT/C. Fig.4 (right) shows what happens when we change the resistance R: if R is increased, the spectral density at low frequencies proportionally increases, but the pole frequency decreases of the same amount. The result is that the area below the curve (i.e. the integral) does not change. The inverse happens if the resistance is decreased). Therefore the kT/C noise depends only on the capacitor value (and, of course, temperature).

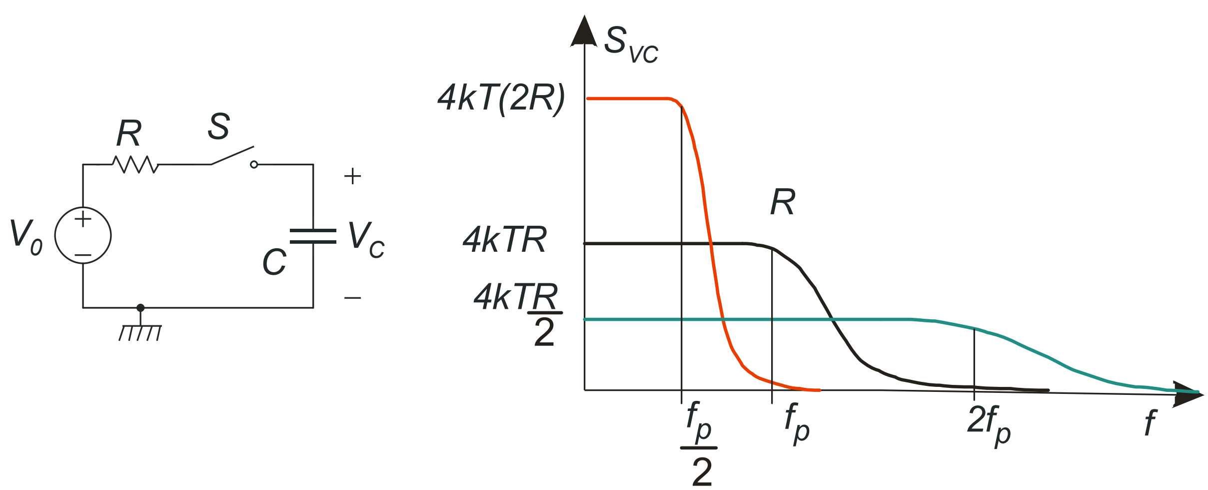

Fig.4

Four important observations can be made at this point:

- The circuit of Fig.4 (left) may represent an equivalent circuit, i.e. a Thevenin equivalent of the network seen from the capacitor terminals of capacitor C. That network can be much more complex than in the simple example of Fig. 4.
- In many cases, it is interesting to consider the charge that is stored into C after each sampling operation. In particular, we are interested in the charge variations with respect to the average, which is equal to CV_0. We can obtain the charge variations (i.e. the charge noise) by simply multiplying the voltage noise ve by C. In terms of mean square value we have to multiply the mean quare value of the voltage by C2. Then:

$$< Q_\varepsilon^2 > = \frac{kT}{C} C^2 = kTC \qquad (12)$$

- The *kT/C* noise represents the minimum noise that is present on the sampled voltage. If the voltage to be sampled (V_0) is affected by other sources of noise, *kT/C* noise is combined to the other sources to form the total sampled noise. This occurs, for example, if we are sampling the output noise of an amplifier that is generally accompanied by a large amount of “excess” noise, which is additional noise with respect to the (unavoidable) thermal noise associated to its output resistance. In this case, the spectral densities of the two noise sources cannot be simply added but there is a sort of mixing that is not simple to model. However, the total noise cannot be smaller than the *kT/C* limit.
- One could argue that this phenomenon does not affect ideal voltage sources, as that shown in Fig.2, since they do not have an internal resistance that produces thermal noise. Actually, we can consider an ideal source as the limit of a real source when *R* tends to zero. Since for whatever small value of *R* the noise mean square voltage is still *kT/C*, then also the limit should be *kT/C*. Considering Fig.4, when *R* tends to zero the PSD gets lower and lower, but its bandwidth (e.g. f_p) extend to infinity, leaving the integral unchanged.

Inclusion of the offset into the noise power spectral density

Figure 5 shows a frequently used simplified representation of noise spectra. This is only a symbolic representation since, in real cases, flicker noise diverges to infinity, when *f* tends to zero; furthermore, the upper band limit (*B*) is not as abrupt as in the figure, but more progressive.

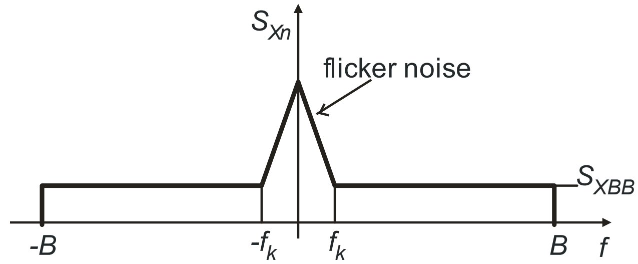

Fig. 5

Nevertheless, this bilateral representation can be very useful to understand what happens to noise in modulation and sampling operations, where the spectra are replicated and shifted along the frequency axis. In order to extend this convenient representation to the other component of additive errors, i.e. offset, we can write the total error:

$$x_{ntot} = x_n + x_{io} \tag{13}$$

Note that even offset can be considered a stochastic process, consisting in a DC signals with random (DC) value. The autocorrelation function of the total additive error is given by:

$$R_{Xntot}(\tau) = < X_{ntot}(t) X_{ntot}(t-\tau) > = < (x_n(t) + x_{io})(x_n(t-\tau) + x_{io}) > \tag{14}$$

Thus:

$$R_{Xntot}(\tau) = < x_n(t) x_n(t-\tau) > + < x_{io}^2 > + < x_{io} x_n(t-\tau) > + < (x_n(t) x_{io}) > \tag{15}$$

Since offset and noise derives from different phenomena, they can be considered independent. Therefore we obtain the following result:

$$R_{Xntot}(\tau) = R_{Xn}(\tau) + \sigma_{Xio}^2 \tag{16}$$

where $R_{Xn}(\tau)$ is the noise correlation function, while σ_{Xio} is the offset standard deviation.

In terms of spectral density, the offset contribution becomes a Dirac delta function with value σ_{Xio}^2. Then we obtain the following symbolic representation that takes into account noise and offset together:

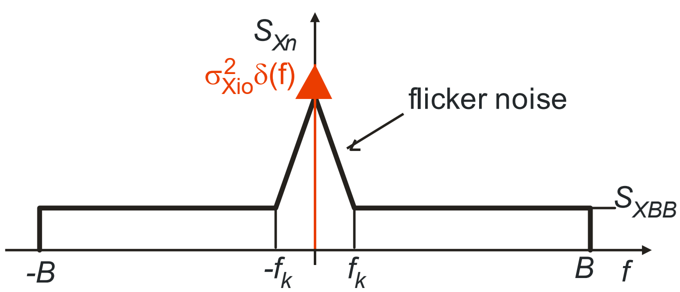

Fig 6