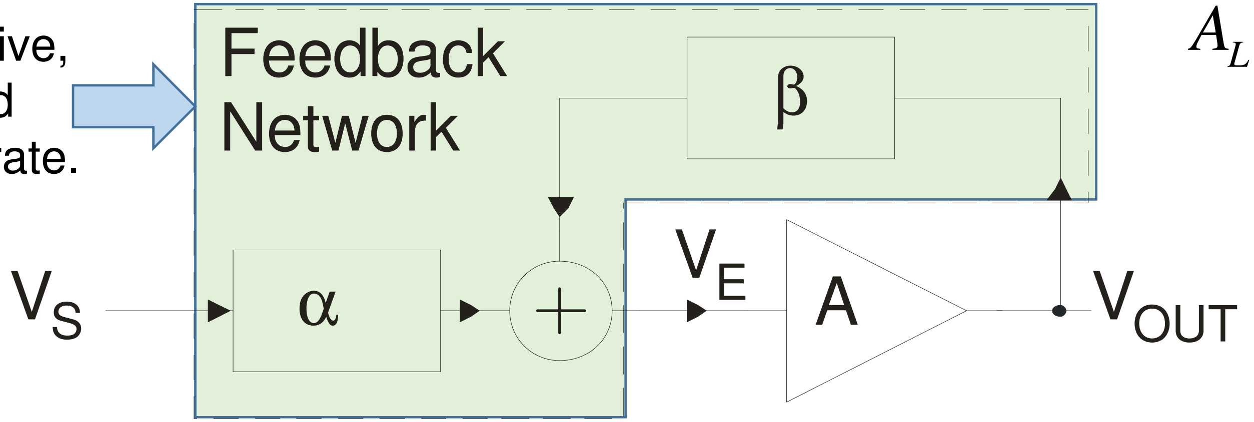
# Use of Pellegrini's cut-insertion theorem for the design of feedback systems

## Block diagram of a feedback system

Typically passive, then linear and possibly accurate.




$$A_L \equiv \frac{V_{OUT}}{V_S} = -\frac{\alpha}{\beta}\frac{\beta A}{\beta A - 1}$$

$$A_{L-ideal} \cong -\frac{\alpha}{\beta}$$

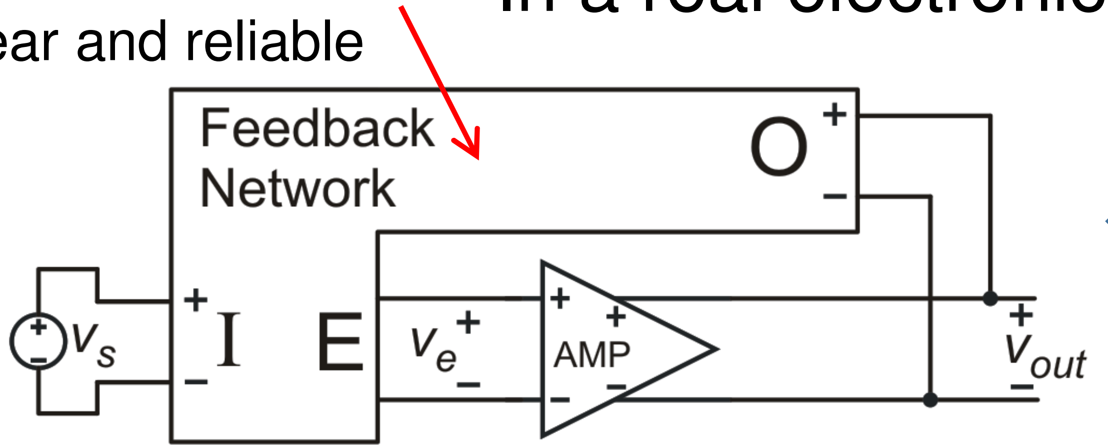
$$\Delta A_L = A_L - A_{L-ideal}$$

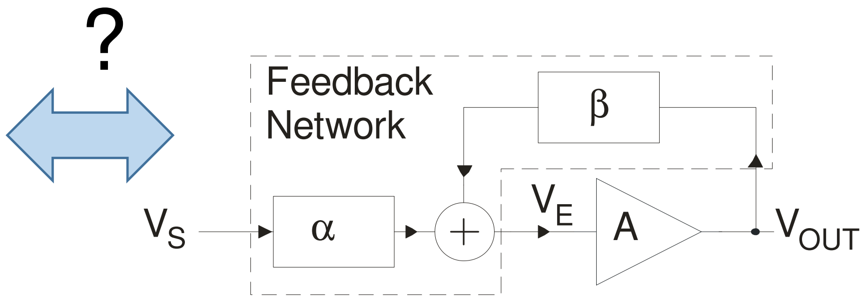
$$\Delta A_L = -\frac{\alpha}{\beta}\left(\frac{\beta A}{\beta A - 1}\right) + \frac{\alpha}{\beta} = -\frac{\alpha}{\beta}\left(\frac{\beta A}{\beta A - 1} - 1\right) = -\frac{\alpha}{\beta}\left(\frac{1}{\beta A - 1}\right) \cong -\frac{\alpha}{\beta}\frac{1}{\beta A} \qquad (|\beta A| >> 1)$$

$$\left|\varepsilon_{AL}\right| \equiv \left|\frac{\Delta A_L}{A_L}\right| \cong |\beta A|^{-1}$$

# In a real electronic network:

Passive network: linear and reliable



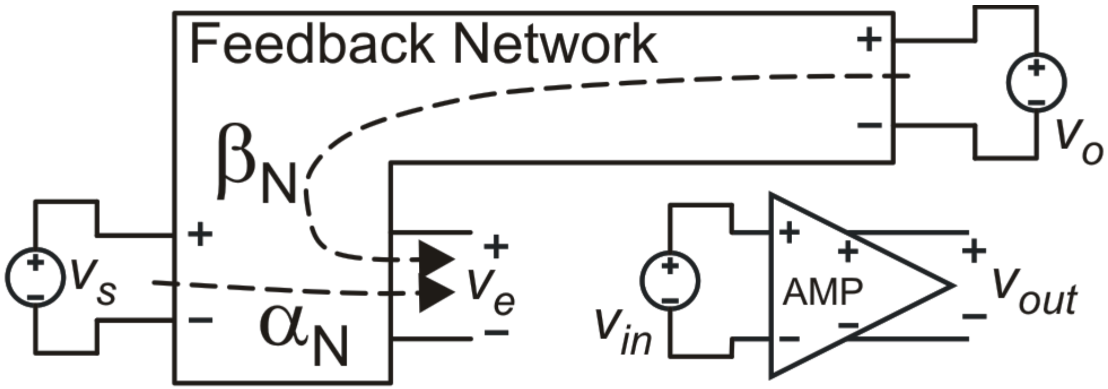




Desired correspondence

$$\alpha_N \to \alpha$$

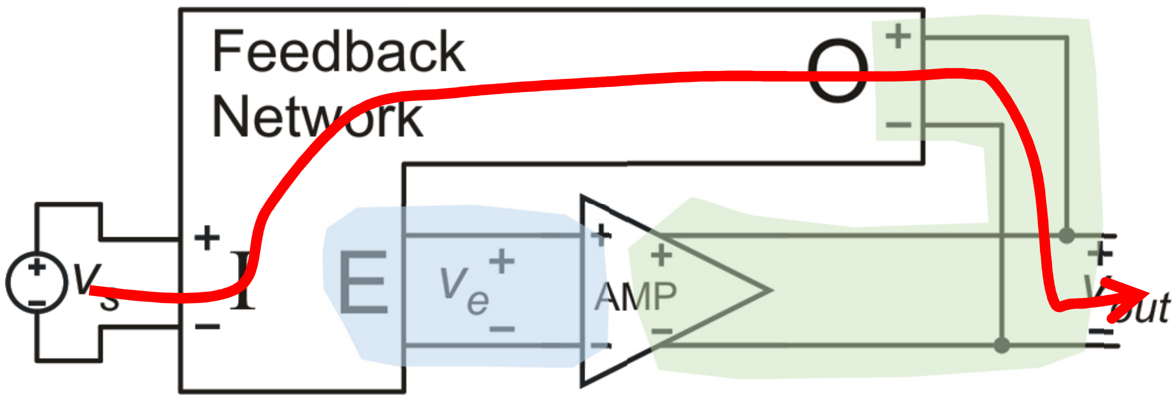
$$\beta_N \to \beta$$

$$A_{OL} = \frac{V_{out}}{V_{in}} \to A$$



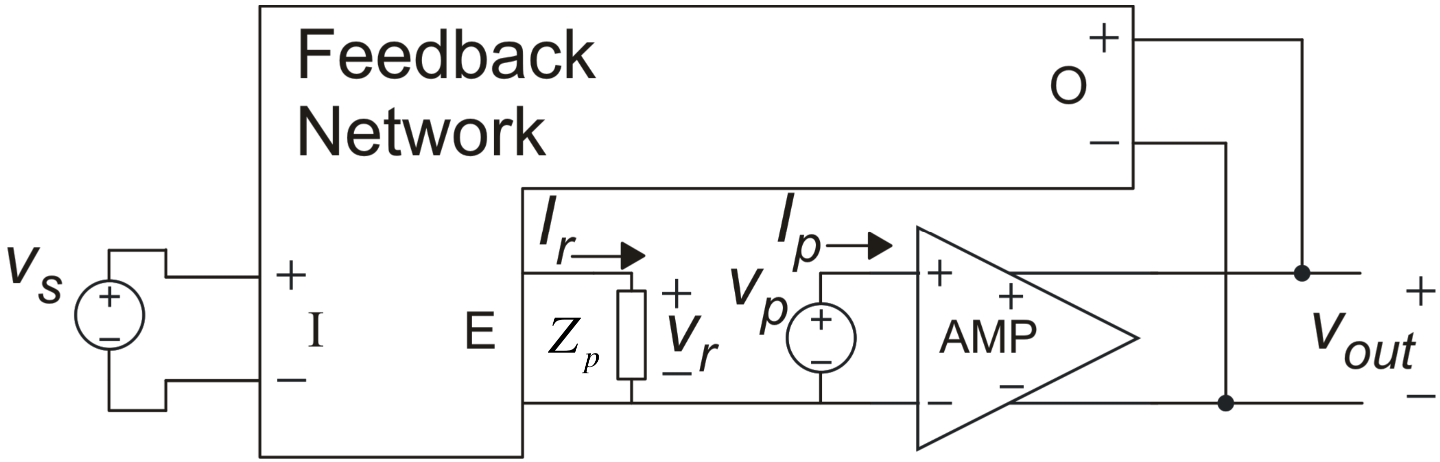

Separation of feedback network and gain element (AMP)

Problems that prevent a direct correspondence between the electrical circuit and the simplified block diagram of a feedback system




- Loading effects of the feedback network on AMP
- Loading effect of the amplifier input impedance on the feedback network
- Direct signal path from the input signal $V_S$ and the output voltage through the feedback network (occurs if the amplifier has a non-zero output impedance).

# Pellegrini's cut-insertion theorem




$A, \beta, \alpha$

now are calculated taking into account loading effects

$\gamma$

takes into account the feed-forward path through the feedback network

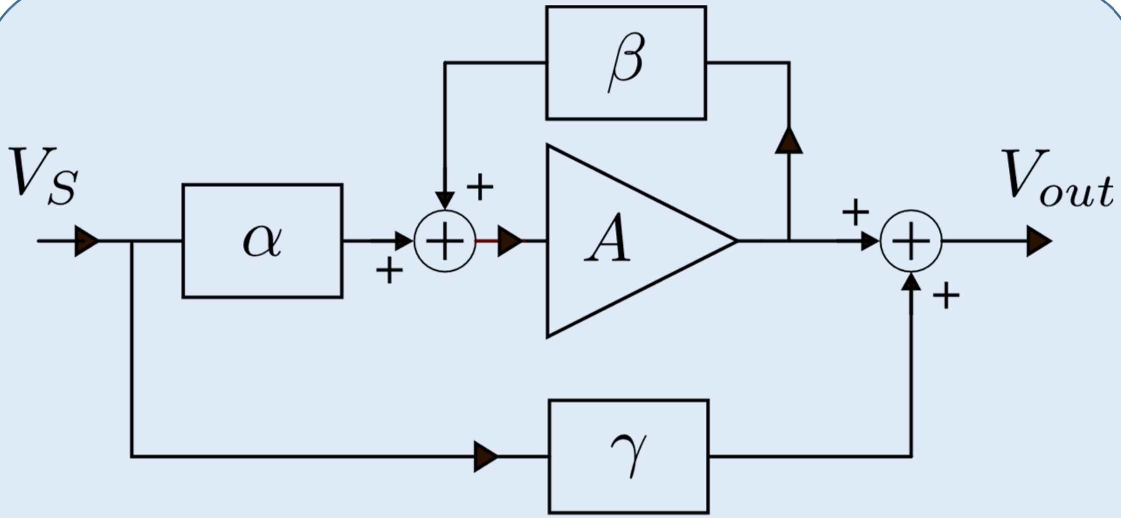
$$Z_i = \left(\frac{v_p}{i_p}\right)_{v_s=0} \; ; \; \beta \equiv \left(\frac{v_r}{v_{out}}\right)_{v_s=0} \; ; \; A = \left(\frac{v_{out}}{v_p}\right)_{v_s=0}$$

$$\frac{1}{Z_p} = \frac{1}{Z_i} + \frac{\rho}{\alpha}(1-\beta A)$$

$$\gamma = \left(\frac{v_{out}}{v_s}\right)_{v_p=0} \; ; \; \alpha \equiv \left(\frac{v_r}{v_s}\right)_{v_p=0} \qquad \rho = \left(\frac{i_p}{v_s}\right)_{v_p=0}$$

ρ is zero if the op-amp is unidirectional

# Block diagram corresponding to the cut-insertion schematization

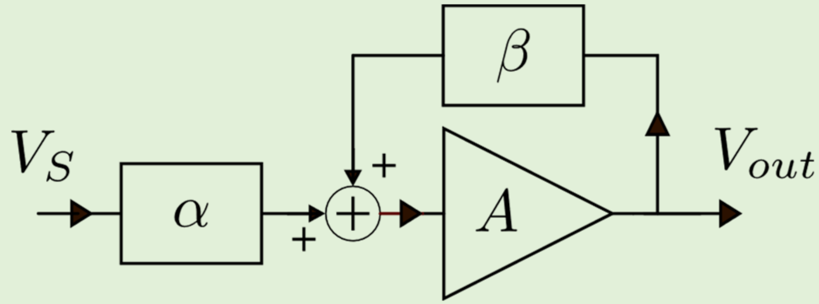



$$\frac{v_{out}}{v_s} = A_L = -\frac{\alpha}{\beta}\frac{\beta A}{\beta A - 1} + \gamma$$

$$\lim_{|\beta A| \to \infty} A_L = -\frac{\alpha}{\beta} + \gamma$$

$$\frac{v_{out}}{v_s} = A_L = -\frac{\alpha}{\beta}\frac{\beta A}{\beta A - 1}$$

$$A_{L-ideal} = \lim_{|\beta A| \to \infty} A_L = -\frac{\alpha}{\beta}$$



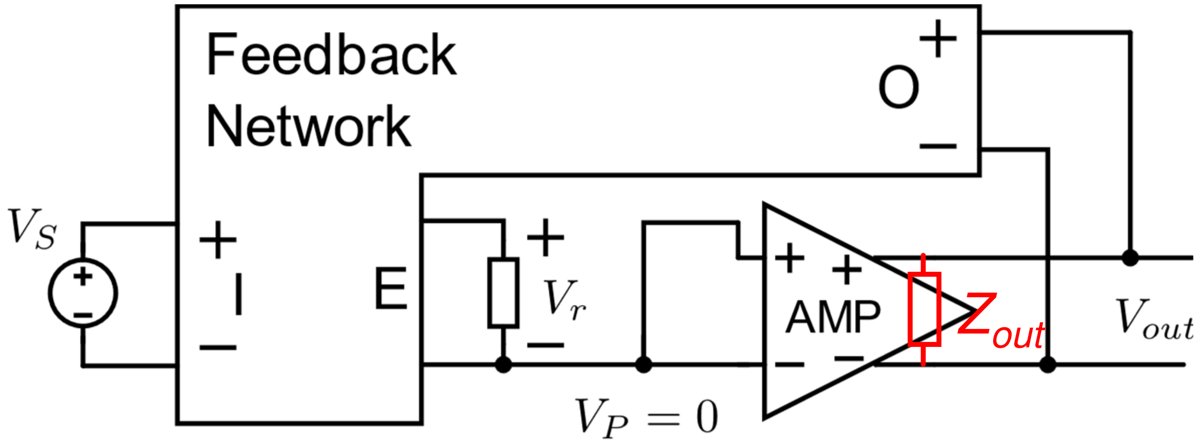

Ideal feedback system

## Problems when using the cut-inserion theorem for design purposes

1. I have to design the feedback network for three transfer functions ($\alpha$, $\beta$ and $\gamma$) instead of only two.
2. $\alpha$ and $\gamma$ depend on the output impedance of the amplifier, $Z_{out}$, which is a parameter that varies much and cannot be predicted reliably.
3. Only in the case that $Z_{out}$ is much smaller than the typical impedances of the feedback network, we can design the network for $\alpha$, $\beta$ and $\gamma$) with port "O" short circuited ($\gamma$=0).

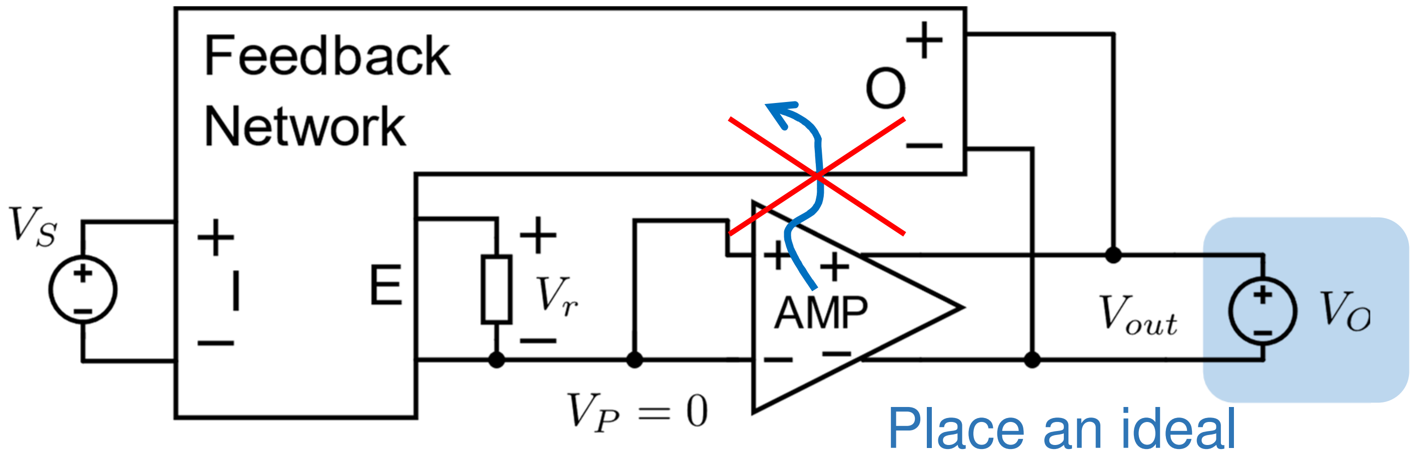
Typically, this does not occur in integrated circuits.

$$\frac{v_{out}}{v_s} = -\frac{\alpha}{\beta} + \gamma$$




Configuration for $\alpha$, $\gamma$

# Cut insertion theorem with modified definition of α (and β)




Place an ideal voltage source at the output

If AMP communicates with the feedback network only through the input and output ports (no intermediate signal path exists between them)

……

then $\beta^* = \beta$

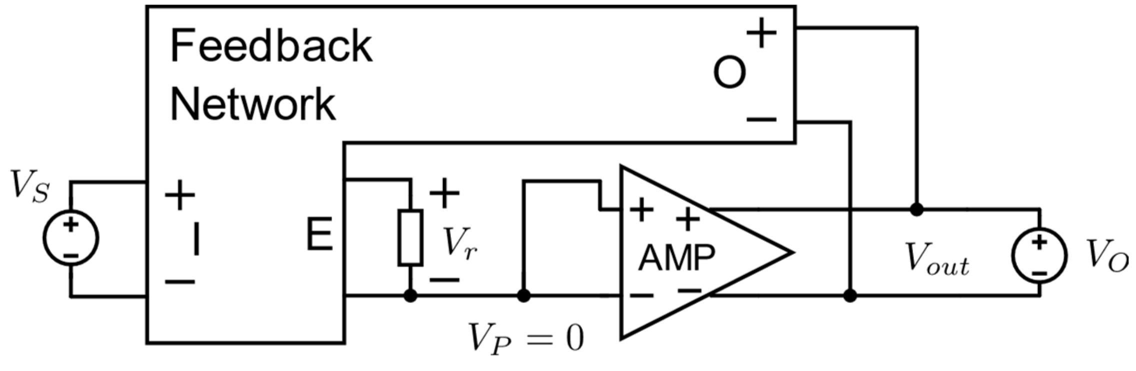
and:

$$\frac{v_{out}}{v_s} = -\frac{\alpha^*}{\beta}\frac{\beta A}{\beta A - 1} + \frac{\gamma}{1 - \beta A}$$

Introduce a new definition of α and β (renamed α* and β*)

$$\alpha^* \equiv \left(\frac{v_r}{v_s}\right)_{v_p, v_o = 0} \quad ; \quad \beta^* \equiv \left(\frac{v_r}{v_o}\right)_{v_p, v_s = 0}$$

# Cut insertion theorem with modified definition of α




$$\frac{v_{out}}{v_s} = -\frac{\alpha^*}{\beta}\frac{\beta A}{\beta A - 1} + \frac{\gamma}{1 - \beta A}$$

Now γ is divided by the loop gain βA

$$\lim_{|\beta A| \to \infty} \frac{v_{out}}{v_s} = -\frac{\alpha^*}{\beta} = -\frac{\alpha^*}{\beta^*}$$

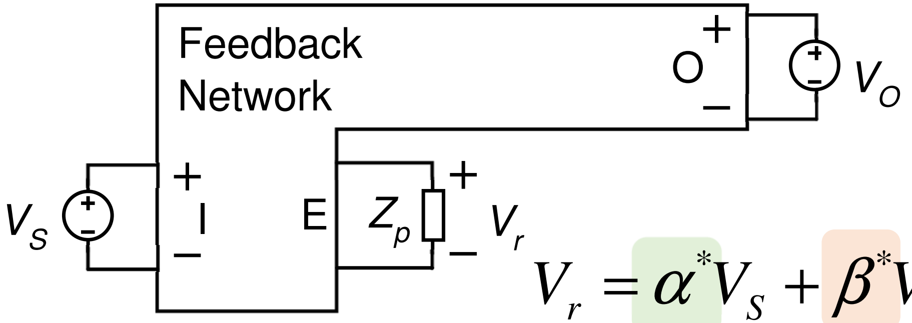
The definition of α* is deeply different from that of α.

Also the  definition of β* is different from that of β. but the two values coincides in most practical cases.

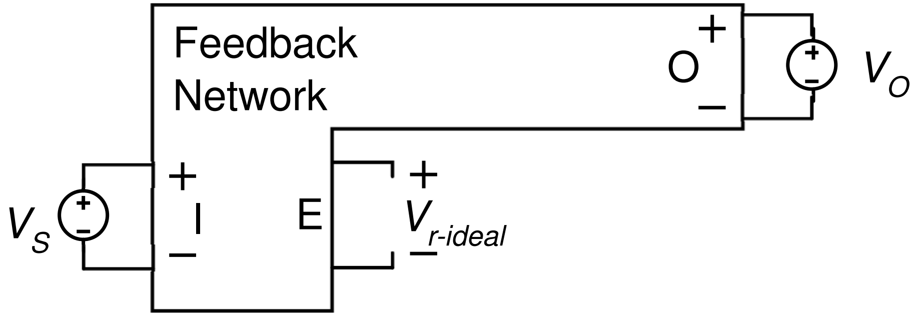
α* and β* are very similar to $\alpha_N$ and $\beta_N$ that are the functions for which we design the feedback network

# Ideal transfer function and network functions

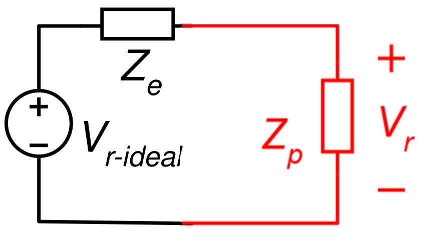
$$\lim_{|\beta A|\to\infty} \frac{v_{out}}{v_s} = -\frac{\alpha^*}{\beta} = -\frac{\alpha^*}{\beta^*}$$




$$V_r = \alpha^* V_S + \beta^* V_O$$




$$V_{r-ideal} = \alpha_N V_S + \beta_N V_O$$




Thevenin equivalent of port E

$$V_r = (\alpha_N V_S + \beta_N V_O)\frac{Z_p}{Z_p + Z_e}$$

$$V_r = \alpha_N \frac{Z_p}{Z_p + Z_e} V_S + \beta_N \frac{Z_p}{Z_p + Z_e} V_O$$

$$\alpha^* = \alpha_N \frac{Z_p}{Z_p + Z_e}$$

$$\beta^* = \beta_N \frac{Z_p}{Z_p + Z_e}$$

$$\frac{\alpha^*}{\beta^*} = \frac{\alpha_N}{\beta_N}$$

# Finite gain error

$$A_{L-ideal} = -\frac{\alpha^*}{\beta^*} = -\frac{\alpha_N}{\beta_N}$$

$$A_L = -\frac{\alpha^*}{\beta}\frac{\beta A}{\beta A - 1} + \frac{\gamma}{1-\beta A}$$

$$\Delta A_L = A_L - A_{L-ideal}$$

$$\Delta A_L = -\frac{\alpha^*}{\beta}\frac{\beta A}{\beta A - 1} + \frac{\gamma}{1-\beta A} - \left(-\frac{\alpha^*}{\beta}\right)$$

$$\Delta A_L = -\frac{\alpha^*}{\beta}\left(\frac{\beta A}{\beta A - 1} - 1\right) + \frac{\gamma}{1-\beta A}$$

$$\Delta A_L = -\frac{\alpha^*}{\beta}\left(\frac{1}{\beta A - 1}\right) + \frac{\gamma}{1-\beta A}$$

$$\frac{\Delta A_L}{A_{L-ideal}} = \left(\frac{1}{\beta A - 1}\right)\left(1 + \frac{\gamma}{A_{L-ideal}}\right)$$

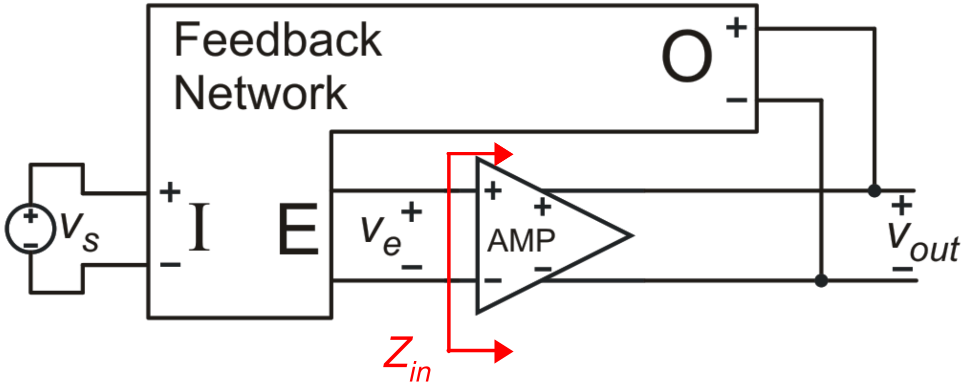
$$e_r = \left|\frac{\Delta A_L}{A_{L-ideal}}\right| \cong \frac{1}{|\beta A|}\left|1 + \frac{\gamma}{A_{L-ideal}}\right|$$

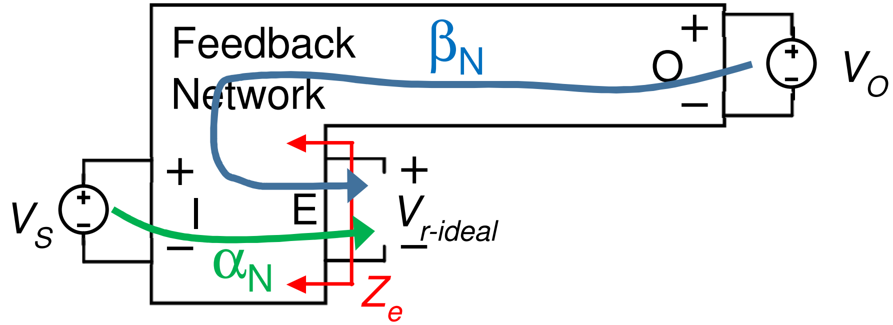
From the Schwarz inequality:

$$e_r \le \frac{1}{|\beta A|}\left(1 + \left|\frac{\gamma}{A_{L-ideal}}\right|\right)$$

# Final recap

$$A_{L-ideal} = -\frac{\alpha^*}{\beta^*} = -\frac{\alpha_N}{\beta_N}$$




$$e_r \leq \frac{1}{|\beta A|}\left(1 + \left|\frac{\gamma}{A_{L-ideal}}\right|\right)$$

$$\beta = \beta^* = \beta_N \frac{Z_{in}}{Z_{in} + Z_e}$$

$$A = A_{VOL} \frac{Z_o}{Z_o + Z_{out}}$$