Modulation of stochastic processes

Let us consider a stochastic process $x_n(t)$ and a modulation signal $y_m(t)$. The target is determining the power spectral density (PSD) of the stochastic process z(t), obtained by multiplication of x_n(t) by y_m(t).

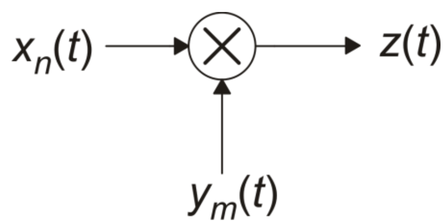

Fig.1 Modulation of a stochastic process x_n(t) by a modulation signal y_m(t).

Case 1: y_m is a simple sinusoidal function (pure tone).

$$y_m(t) = A\cos(\omega t) \tag{1}$$

In this case, the output process $z(t)$ is not stationary, since, for example, its mean square value $<z^2(t)>$ depends on t. In order to obtain a stationary process (necessary condition for the existence of the PSD), it is necessary that also y_m is a stochastic process. This can be obtained by introducing a random phase to the signal in (1), obtaining:

$$y_m(t) = A\cos(\omega t + \theta) \tag{2}$$

where θ is uniformly distributed in the interval [0-2π]. This represent the real case, since there is no reason for the existence of a phase relationship (i.e. synchronization) between x_n and y_m.
The autocorrelation function R_{ZZ} is given by:

$$R_{ZZ}(\tau) =< z(t)z(t-\tau) > \; = \; < x_n(t)A\cos(\omega t+\theta)x_n(t-\tau)A\cos[\omega(t-\tau)+\theta]> \tag{3}$$

Grouping the terns in x_n:

$$R_{ZZ}(\tau) =< z(t)z(t-\tau) > \; = \; < x_n(t)x_n(t-\tau)A^2\cos(\omega t+\theta)\cos[\omega(t-\tau)+\theta]> \tag{4}$$

Since the process x_n is independent of the variable θ, the average can be split into the product of two distinct averages:

$$R_{ZZ}(\tau) =< z(t)z(t-\tau) > \quad = \quad A^2 < x_n(t)x_n(t-\tau) > \cdot < \cos(\omega t+\theta)\cos[\omega(t-\tau)+\theta]> \tag{5}$$

The first average is simply the autocorrelation function of $x_n(t)$, $R_{xx}(\tau)$.
To calculate the second average, it is convenient to transform the cosine product using the identity:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)] \tag{6}$$

obtaining:

$$< \cos(\omega t+\theta)\cos[\omega(t-\tau)+\theta]> \; = \; < \frac{1}{2}[\cos(2\omega t-\omega\tau+\theta)+\cos(\omega\tau)]> \tag{7}$$

For the linearity of the average operator:

$$<\cos(\omega t+\theta)\cos[\omega(t-\tau)+\theta]> \;=\frac{1}{2}[<\cos(2\omega t-\omega\tau+2\theta)>+<\cos(\omega\tau)>] \tag{8}$$

It can be easily shown that: $<\cos(2\omega t-\omega\tau+2\theta)> = 0$ since for any value of t and τ, the average obtained by varying q across the whole [0-2π] interval is zero.
Thus:

$$<\cos(\omega t+\theta)\cos[\omega(t-\tau)+\theta]> \;=\frac{1}{2}\cos(\omega\tau) \tag{9}$$

Finally, we obtain:

$$R_{ZZ}(\tau)=R_{XX}(\tau)\frac{A^2}{2}\cos(\omega\tau) \tag{10}$$

In the frequency domain, we obtain the following relationship between the PSDs:

$$S_Z(f)=\frac{A^2}{4}[S_{Xn}(f-f_m)+S_{Xn}(f+f_m)] \tag{11}$$

where $f_m=\omega/2\pi$ is the modulation frequency. Graphically, the operation is depicted in Fig.2.

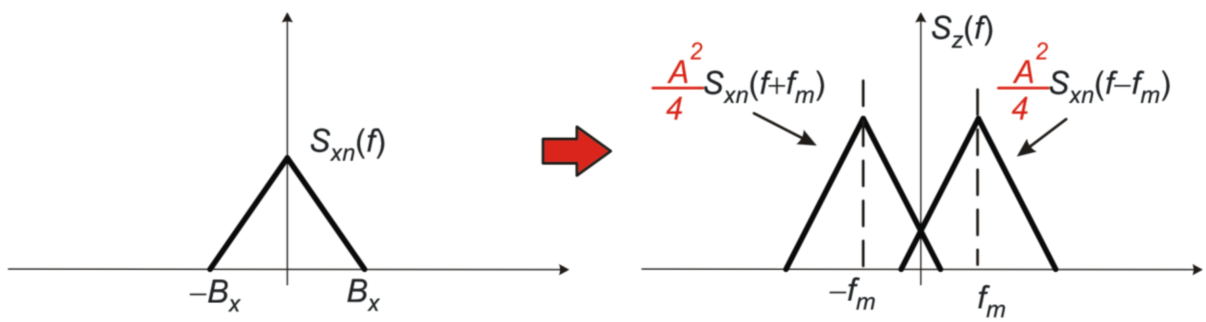

Fig.2 Effect of the modulation of a stochastic process, of PSD S_{Xn}, by a simple cosine function.

Case 2: y_m is a sum of sinusoidal functions:

$$y_m(t)=\sum_{i=1}^{n}A_i\cos(\omega_i t+\theta_i) \tag{12}$$

with A_i, ω_i and θ_i known values (i.e. they are not random variables). Again, the expression in (12) is a deterministic signal and, as such, it is not suitable to produce a stationary process $z(t)$. Then we introduce a random delay, t_R, representing the fact that there is no synchronization between $x_n(t)$ and $y_m(t)$. Then, the modulation signal that we will consider is:

$$y_m(t)=\sum_{i=1}^{n}A_i\cos[\omega_i(t-t_R)+\theta_i] \tag{13}$$

We can repeat the passages made for the single sinusoid up to (4), exploiting the independence of x_n and y_m, obtaining:

$$R_{ZZ}(\tau) = R_{XX}(\tau) < y_m(t) y_m(t-\tau) > \tag{14}$$

$$y_m(t) y_m(t-\tau) = \sum_{i=1}^{n} A_i \cos[\omega_i (t - t_R) + \theta_i] \sum_{i=1}^{n} A_i \cos[\omega_i (t - \tau - t_R) + \theta_i] \tag{15}$$

$$y_m(t) y_m(t-\tau) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ A_i A_j \cos[\omega_i (t - t_R) + \theta_i] \cos[\omega_j (t - \tau - t_R) + \theta_j] \right\} \tag{16}$$

Transforming the cosine product, we obtain:

$$\begin{aligned} y_m(t) y_m(t-\tau) = & \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{A_i A_j}{2} \cos[(\omega_i + \omega_j)(t - t_R) - \omega_j \tau + \theta_i + \theta_j] \right\} + \\ & + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ \frac{A_i A_j}{2} \cos[(\omega_i - \omega_j)(t - t_R) + \omega_j \tau + \theta_i - \theta_j] \right\} \end{aligned} \tag{17}$$

Applying the average operator to this expression and exploiting linearity, all terms where a dependence on t_R is present produce a zero average, since t_R varies uniformly across a very wide interval, ideally spanning from -∞ to ∞.
The only terms that do not produce a zero contribution to the total average are the ones that contain the difference ω_i-ω_j, for the particular case $i=j$, since the dependence on t_R is cancelled. With simple passages:

$$< y_m(t) y_m(t-\tau) >= \sum_{i=1}^{n} \frac{A_i^2}{2} \cos(\omega_j \tau) \tag{18}$$

Finally, from (14) and (18) we obtain:

$$R_{ZZ}(\tau) = R_{XX}(\tau) \sum_{i=1}^{n} \frac{A_i^2}{2} \cos(\omega_j \tau) \tag{19}$$

Equation (19) shows that the effect of modulating the random x_n process by a sum of sinusoidal functions at different frequencies, correspond to multiplication of the autocorrelation function by a sum of cosine functions. The phases, θ_i , do not have any effect on the resulting PSD.
In the frequency domain, (19) is represented by Fig.3, where an example of modulating signal composed by the sum of three sinusoidal waveform is considered. The coefficients placed close to each replica are the numbers that multiply the replicas.

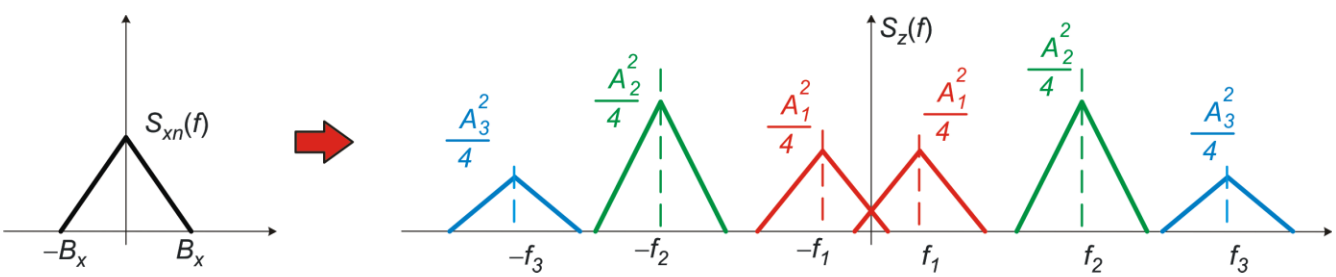

Fig.3 Effect of the modulation of a stochastic process, of PSD S_{Xn}, by a sum of three sinusoidal functions of frequencies $f_i = \omega_i / 2\pi$ and peak value A_i, with i=1,2,3.

Frequently it is more convenient to express the sum in (12) with the complex notation:

$$y_m(t) = \sum_{i=-n}^{n} C_i e^{j\omega_i t} \tag{20}$$

where C_i are complex numbers related to A_i and θ_i by:

$$C_i = \frac{A_i}{2} e^{j\theta_i} \;\; ; \quad C_{-i} = C_i^* \Rightarrow |C_i|^2 = |C_{-i}|^2 = \frac{A_i^2}{4} \tag{21}$$

Considering (21), it is possible to express the multiplying coefficients of the replicas as a function of C_i instead of A_i, obtaining Fig.4, which is analogue to Fig.3.

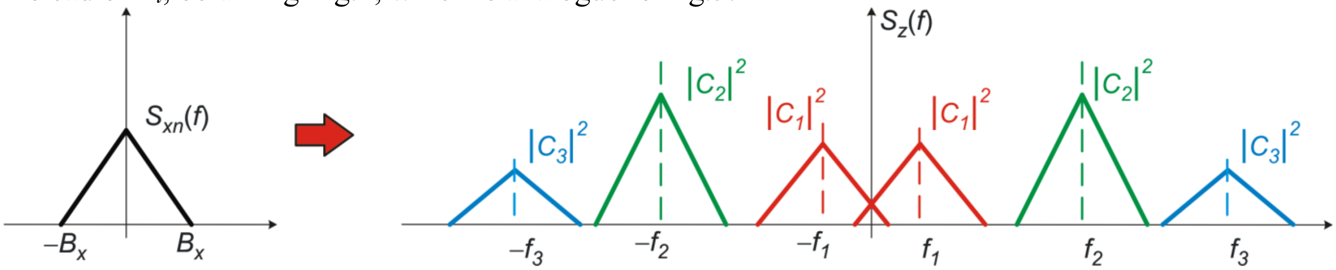

Fig.4 Effect of the modulation of a stochastic process, of PSD S_{Xn}, by a sum of three sinusoidal functions of frequencies $f_i=\omega_i/2\pi$ and complex coefficients C_i, with i=-3,-2,-1, 1, 2, 3.

A particular case of modulating signal is the periodic waveform. A periodic waveform (e.g. a square waveform) can be expressed in the form given by (20), with the additional condition that all angular frequencies are multiple of a particular angular frequency ω_0 (the fundamental). Furthermore, the sum can be infinite and include the zero order term (C_0 , i.e. the DC component). The PSD of a stochastic process modulated by a periodic signal is obtained in the same way as the signal in (20), therefore it can be represented by Fig.4. The DC component, if present, gives a replica placed around the origin.

Relationships between discrete time (DT) and continuous time (CT) signals.

DT signals assume a significant value only at instants that constitute a numerable set. Typically, the instants are evenly spaced, so that two successive instants are always separated by the same "sampling time", that will be indicated with T. In particular:

T= Sampling time
f_S=1/T= Sampling frequency.

It is important to observe that a DT signal simply does not exist at instants different from the sampling instants. A DT signal can be thought as the result of sampling a CT signal, but this is not necessary and any sequence of values can be considered a DT signal. Nevertheless, since DT signal are used to represent CT signals, it is useful to consider the sampling process.
Fig.5, in the upper part, shows a CT signal $s(t)$ and its spectrum $S(f)$ (Fourier transform). Note that $S(f)$ represents the complex amplitude of the (infinitesimal) exponential into which $s(t)$ can be decomposed. That is:

$$s(t) = \int_{-\infty}^{+\infty} S(f) e^{j2\pi ft} df \tag{22}$$

Note: in order to be able to represent all (finite energy) signals, it is necessary to include into the sum exponentials of frequencies, that, in terms of absolute value, span from 0 to ∞. The lower part of Fig.5 shows a discrete signal obtained by sampling $s(t)$. Since (22) is capable of producing the value of $s(t)$ for every time t, it can also reproduce the values at the sampling instants, i.e. the DT signal:

$$s(nT) = \int_{-\infty}^{+\infty} S(f) e^{j2\pi fnT} df \qquad \text{with } n \text{ integer} \tag{23}$$

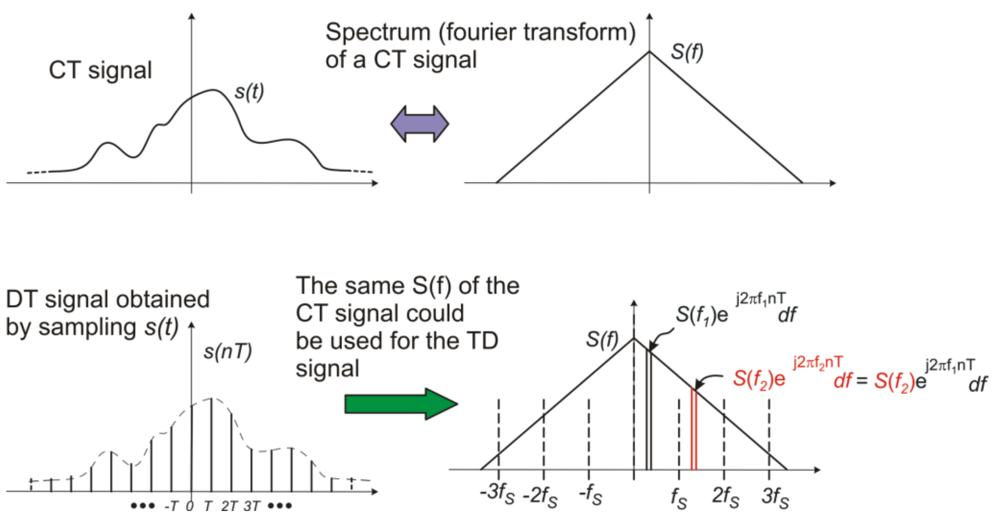

Fig.5 Time domain and frequency domain transformations regarding the creation of a discrete time signal from a continuous time one.

For this reason, we could use $S(f)$ also as a decomposition of $s(nT)$ into an infinite sum (i.e. integral) of DT exponential functions: e^{j2pnT}. But, while this decomposition is unique for $s(t)$, this is not the case for the DT signal. In fact, as the spectrum in the bottom right corner of Fig.5 shows, two DT exponentials of frequencies f_1 and f_2, with $f_2=f_1+f_S$, produce the same identical sequence, thus they are undistinguishable.
Thus, instead of summing two contributions at frequencies f_1 and f_2, we can include into the integral only a contribution at f_1 but with amplitude $(S(f_1)+S(f_2))df$. This is equivalent to summing the spectrum $S(f)$ with a replica shifted by $-f_S$, as shown in Fig.6.

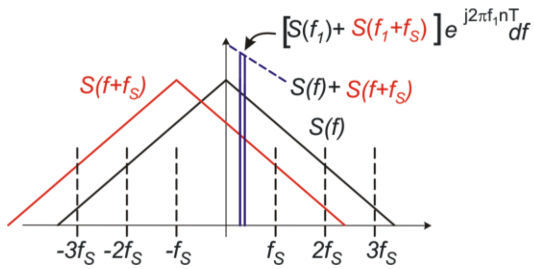

Fig.6. This figure shows how the contributions at frequency f_1 and f_2 can be included into the integral as only one contribution at frequency f_1.

Clearly, the contribution at f_2, included into (23) is now represented by the cumulative contribution at f_1, thus I should not consider f_2 anymore. The same consideration can be repeated for any couple of frequencies whose difference is a multiple of f_S. As a result, I can express the DT signal $s(nT)$ as the integral of exponentials with frequencies restricted only to an interval as large as f_S. These exponential are multiplied by a function that should also include contributions from the whole $[-\infty, +\infty]$ interval, at frequencies that can be represented in the interval of width f_s with the procedure illustrated in Fig.6. This function is called Discrete Fourier Transform (DFT). The DFT of a DT signal obtained by sampling a CT signal can be obtained simply by summing replicas of the original spectrum shifted by all multiples of f_S, as shown in Fig. 7. The sum must be evaluated only into the chosen frequency interval of width f_S. This interval, as shown in Fig.7 is usually $[-f_S/2, f_S/2]$. It can be easily shown that the sum is periodical with period f_S, so that choosing other intervals gives the same behavior.

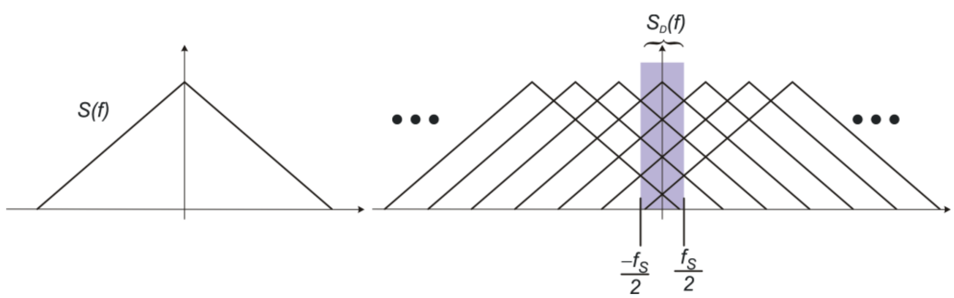

Fig.7 Procedure used to obtain the DFT from the spectrum (Fourier transform) of the original CT signal. Note that the resulting function is defined only in a restricted interval.

Discrete time stochastic processes can be considered as the result of sampling CT stochastic processes with sampling interval T. The resulting DT autocorrelation function turns out to be a sampled version of the CT autocorrelation, with the same period T. Thus, the DT PSD (also indicated as D-PSD), can be obtained with the same procedure used for the DFT of deterministic signals (procedure illustrated in

Fig.7. Note that the mean square value of the DT signal is obtained by integrating its D-PSD only over the [$-f_S/2, f_S/2$] interval.

Continuous time signals deriving from a sampling process.

Sampling can be used also to obtain CT signals. These signals are defined for any time value of the real axis and are then deeply different from DT signals. In particular, their spectrum is defined over the whole frequency axis (i.e from $f=-\infty$ to $f=+\infty$). A couple of signal belonging to this category that can be frequently useful are:

1) Delta-sampled signal: the CT signal deriving from multiplying the original signal with an infinite periodical sequence of unity-area Dirac delta functions ("Dirac comb").
2) Sampled and Held (S&H) signal: the CT signal obtained by sampling the original signal and holding the sampled value until the next sampling instant.

Both signals are CT signals. This is clear for the S&H signal. Delta-sampled signals are often considered DT signals. This not true since the delta-sampled signal is 0 (thus it is defined) everywhere, except at the sampling instants. Note that the signal is not defined in a conventional sense at the sampling intervals, where the delta are located. However, the delta function can be considered the limit of very short rectangular functions, whose duration tends to zero and amplitude to infinity, maintaining a unity area (integral). For this reason, the delta-sampled signal can be considered as the limit of conventional CT signals and then a CT signal itself.
Figure 8 shows the transformations that are produced on the spectrum when a CT signal $s(t)$ is sampled by Dirac deltas and then transformed into a S&H signal. The spectrum of the delta-sampled signal consist in an infinite number of replicas of the original signal. Replicas are shifted by multiples of the sampling frequency f_S : the whole frequency axis should be considered, and not only the result on a restricted interval, since we are dealing with a CT signal. The delta-sampled signal does not generally represents a practical signal; it is used as an intermediate signal to calculate the spectrum of more useful signals, as the sample and held one.
In this case, we can consider that the S&H signal is the result of the convolution of the delta-sampled signal by the rectangular function $h_R(t)$ shown in Fig.8. This correspond to multiplying the spectrum of the delta-sampled signal by the function:

$$T \cdot e^{-j\pi f T}\,\mathrm{sinc}(\pi f T) \tag{24}$$

where the "sinc" function is defined here as:

$$\mathrm{sinc}(x) \equiv \frac{\sin(x)}{x} \tag{25}$$

Then, the spectrum of the sample and held signal can be calculated by summing all the replicas obtained by shifting the original spectrum by multiple of f_S ad then multiplying the resulting spectrum by the function given in (24), without the initial "T" coefficient. The latter is cancelled by the $1/T$ factor applied to the original spectrum due to the transformation into the delta-sampled signal.

In the case of stochastic processes, the same modifications applies to the PSDs, with the only difference that the $1/T$ and T factors are squared. However, they still cancel each other when calculating the spectrum of the sampled and held process, which is then calculated by simply summing the replicas of the original PSD, shifted by multiples of f_S and then multiplying the result by:

$$\left| e^{-j\pi fT} \operatorname{sinc}(\pi fT) \right|^2 = \operatorname{sinc}^2(\pi fT) \quad (26)$$

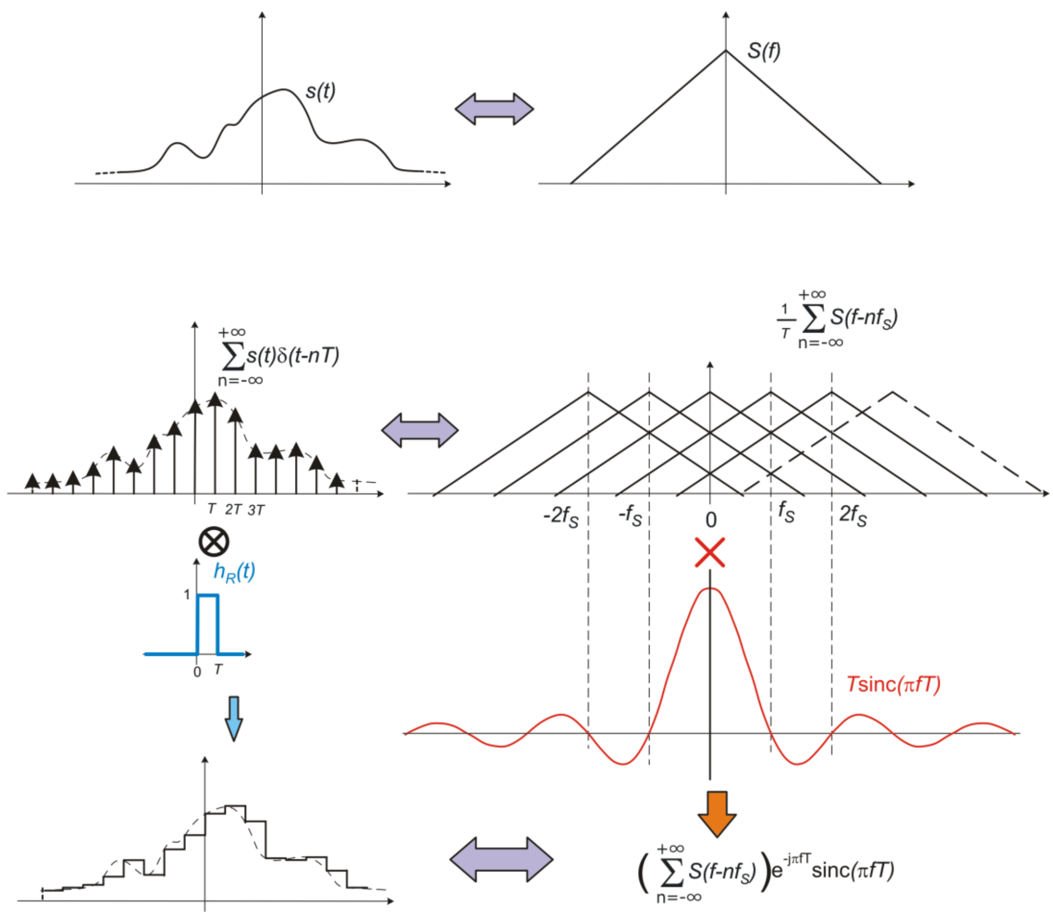

Fig.8 Transformations of a deterministic signal spectrum (Fourier transform) after delta sampling and sample & hold operation.