Analog Filter Design

Part. 5: Discrete time filters

Discrete time (DT) signals and sequences

- A discrete time signal is defined only on a "countable set" of time instants. The set, or time series, can be either finite or infinite.
- Definition of a discrete time signal can also disregard the actual times at which each value corresponds. In this way we have a pure, ordered sequence of values.

Discrete time signal: $x(t_n)$
Pure sequence: $x(n)$

- A discrete time signal **may** be the result of sampling a Continuous Time (CT)signal. Sampling is generally considered to be uniform. Generally, we are interested to DT signals for their capability to represent CT signals.

Discrete Time Signals (DTS): Linear operators

- As with CT signals, while dealing with DT signals we are interested in Linear, Time Invariant, Causal systems.
- In DT signals the derivative operator is substituted by the difference operator:

CT domain DT domain

$$\frac{dx(t)}{dt} \Leftrightarrow x(n) - x(n-1)$$

- More generally, the base operator in DT signal is the unity delay operator:

$$x(n) \Rightarrow x(n-1)$$ " T " operator

Difference Equations

- In the DT domain, differential equations are substituted by difference equations, where difference between elements of the sequences taken with different indexes (e.g. n, n-1, n-2 etc.) appears.

First (Δ) and second (Δ^2) difference definitions (non causal operators)

$$\Delta x(n) = x(n+1) - x(n)$$

$$\Delta^2 x(n) = \Delta x(n+1) - \Delta x(n) = x(n+2) - 2x(n+1) + x(n)$$

- Strictly speaking, difference equations are a particular case of recurrence equations:

$$y(n+1) = f\left[x(n+1), x(n), \ldots. x(n-k), \;\; y(n+1), y(n), \ldots. y(n-k)\right]$$

Analysis tool: Z transform

- In the case of linear, time invariant and causa recurrence equations, a powerful approach is using the Z-transform, which is the analogue of the Laplace transform.

- With the Z-transform, the unity delay operator is transformed into multiplication by Z^{-1} : recurrence equations becomes algebraic equations.

$$x(n) \quad \Rightarrow \quad x(n-1)$$

$$X(z) \quad \Rightarrow \quad Z^{-1}X(z)$$

Common Z-Transform pairs

$$\delta(n) \leftrightarrow 1$$

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$u(n)a^{n} \leftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$u(n)na^{n} \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^{2}} = \frac{az}{\left(z-a\right)^{2}}$$

Note that $u(n)a^n$ is an exponential function:

$$a^{n} = e^{\ln(a)n}$$

Exponential functions are eigenvectors of the delay operator

Z-Transform applied to LTI

LTI (Linear Time Invariant) system representation in the DT domain:

$$Y(z) = H(z)X(z) = \frac{N(z)}{D(z)}X(z)$$

Rational transfer function representations

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots . + a_N z^{N}}$$

Negative powers (preferred for synthesis)

$$H(z) = \frac{b_0 z^{M} + b_1 z^{M-1} + \ldots + b_{M-1} z^{M-1} + b_M}{a_0 z^{N} + a_1 z^{N-1} + \ldots .. + a_{N-1} z^{N-1} + \ldots . + a_N} z^{N-M}$$

Positive powers

Z-Transform: a few properties

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X(z)$$ Final value theorem

$$x(0) = \lim_{z\to\infty} X(z)$$ Initial value theorem

DC gain of a transfer function *H(z)* $= \lim_{z\to 1} H(z)$

$$H(z) = \frac{N(z)}{D(z)} \Leftrightarrow$$ Stability: for all poles (D(z) roots) z_i: $|z_i| \leq 1$

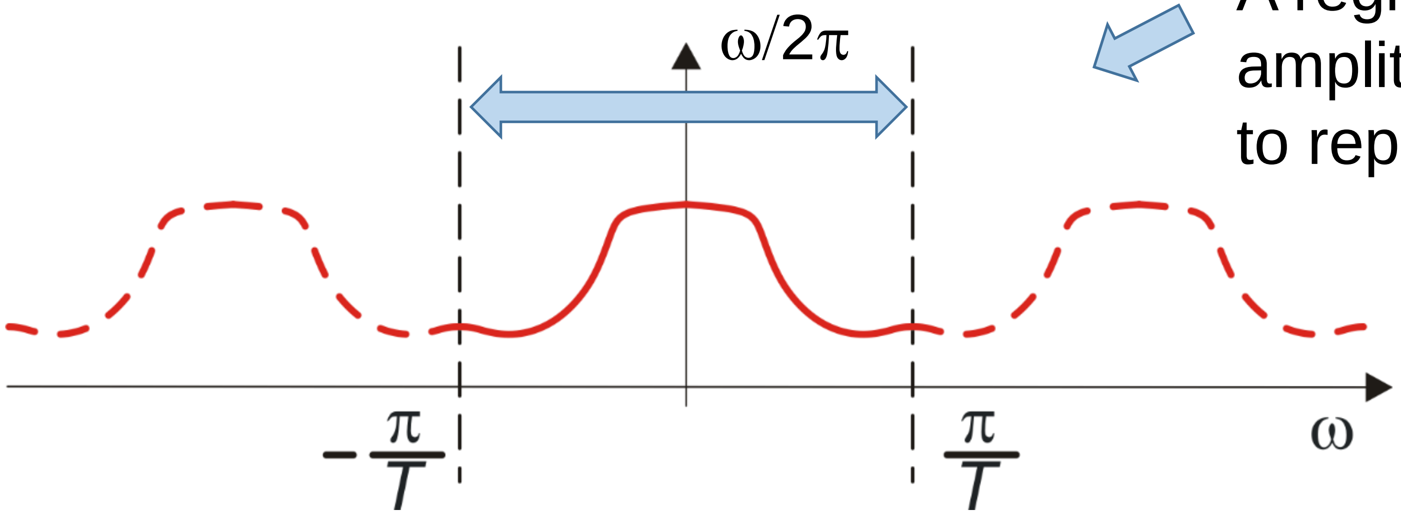
DT signals: Fourier Transform (DTFT)

$$x(nT) = \int_{-\infty}^{+\infty} X_F(\omega) e^{j\omega Tn} d\left(\frac{\omega}{2\pi}\right)$$

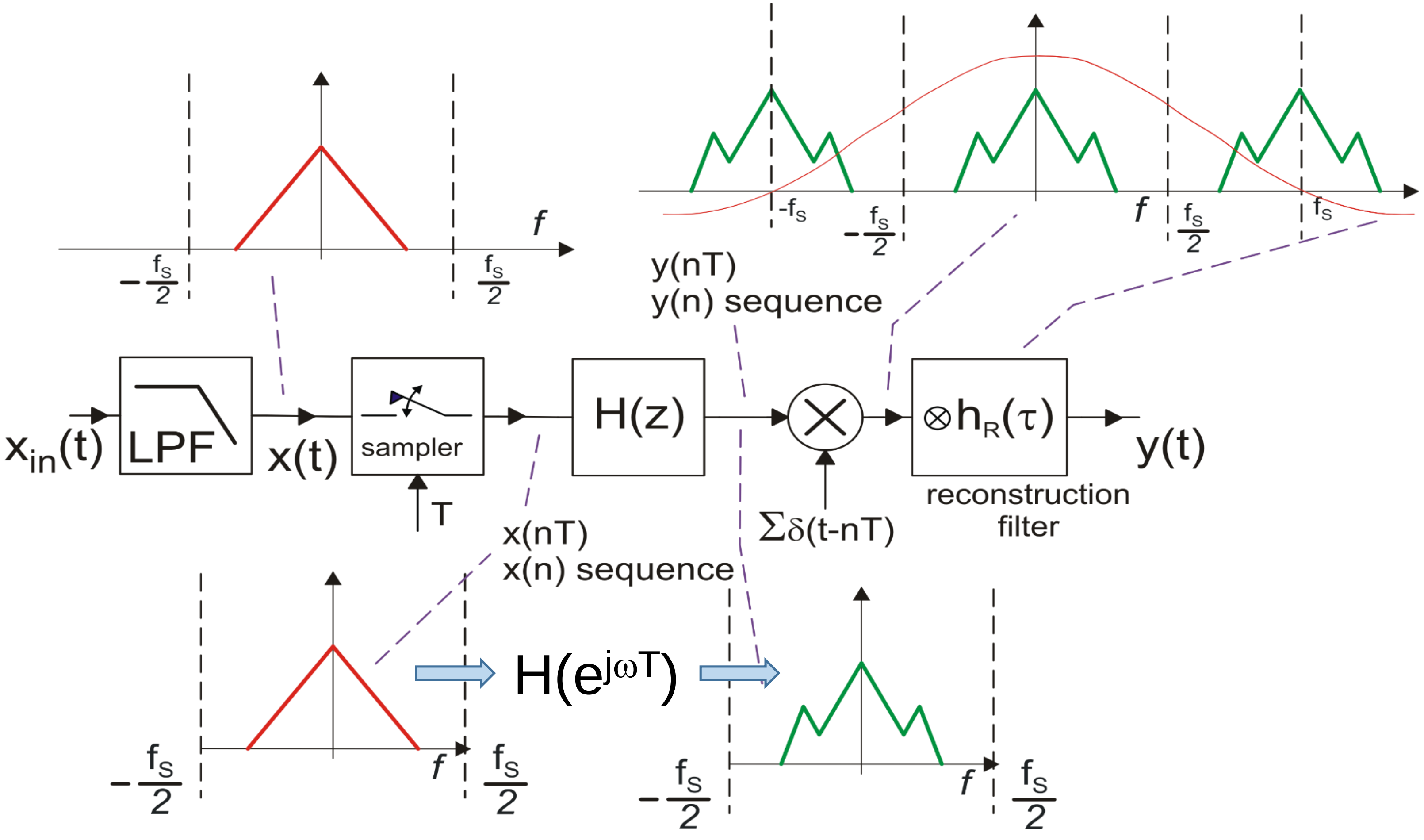
Decomposition of x(nT) into an integral of complex exponential functions

$$\left.\begin{array}{l} \omega T \to \omega T + 2\pi x(nT) \\ \omega \to \omega + 2\pi / T \end{array}\right\} \; e^{j\omega Tn} \to e^{j\omega Tn + j2\pi n} = e^{j\omega Tn}$$

A region of ω / 2π amplitude is sufficient to represent the sequence



DT filters used to replace CT filters



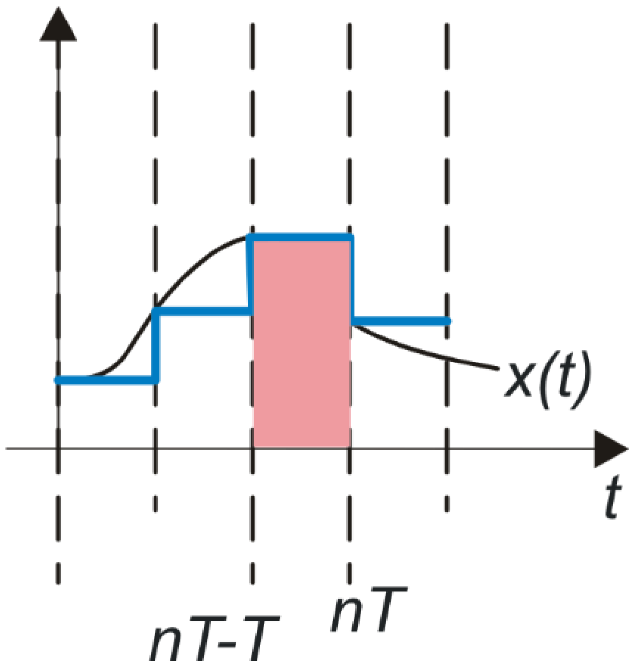
DT Filters synthesis (Ideal block diagrams)

- Start from a CT state variable filter and replace the CT integrator with DT integrator ⇨ IIR

- Start from a CT transfer function, HCT(s), and transform it into a DT transfer function H(z) ⇨ IIR

- Use synthesis approaches that do not need an analog filter as a starting point: use the delayed impulse response properly windowed ⇨ FIR

Method 1: simulate CT state variable filters



$$y(t) = \omega_0 \int_{t_0}^{t} x(\tau) d\tau$$

$$y(nT) = y(nT - T) + \omega_0 T x(nT - T)$$

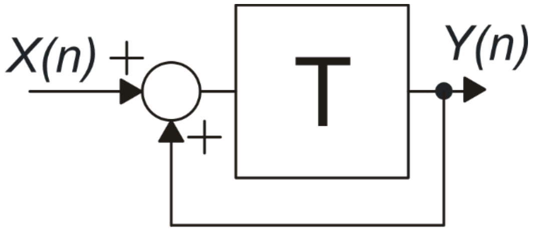
$$y(n) = y(n-1) + T\omega_0 x(n-1)$$

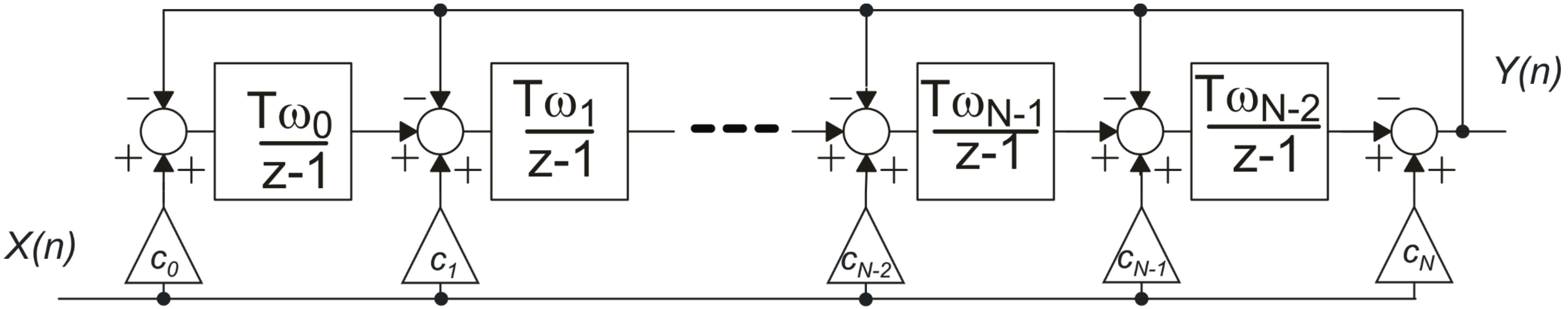
$$Y(z) = z^{-1} Y(z) + T\omega_0 z^{-1} X(z)$$

$$\boxed{H_I(z) = T\omega_0 \frac{z^{-1}}{1 - z^{-1}} = T\omega_0 \frac{1}{z-1}}$$

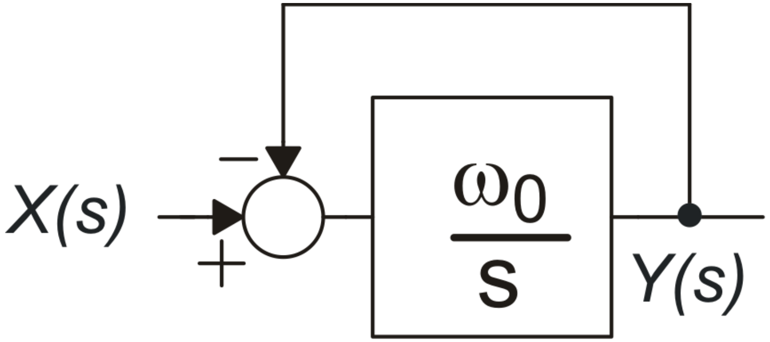
Simulation of CT state variable filters: result

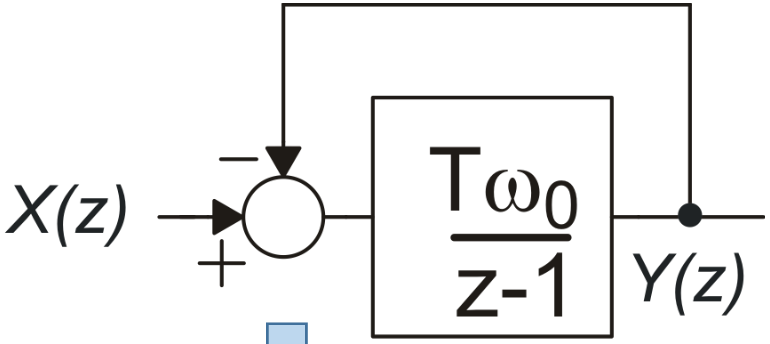
$$\frac{1}{z-1}$$





Example: Transform a 1st order low pass filter





$$H_{CT}(s) = \frac{\omega_0}{s + \omega_0}$$

$$Y(z) = \frac{T\omega_0}{z-1}\left[X(z) - Y(z)\right]$$

$$H(z) = \frac{T\omega_0}{z - 1 + T\omega_0} = \frac{T\omega_0 z^{-1}}{1 + (T\omega_0 - 1)z^{-1}}$$

$$z_p = 1 - T\omega_0$$

$0 < T\omega_0 < 2$ stability

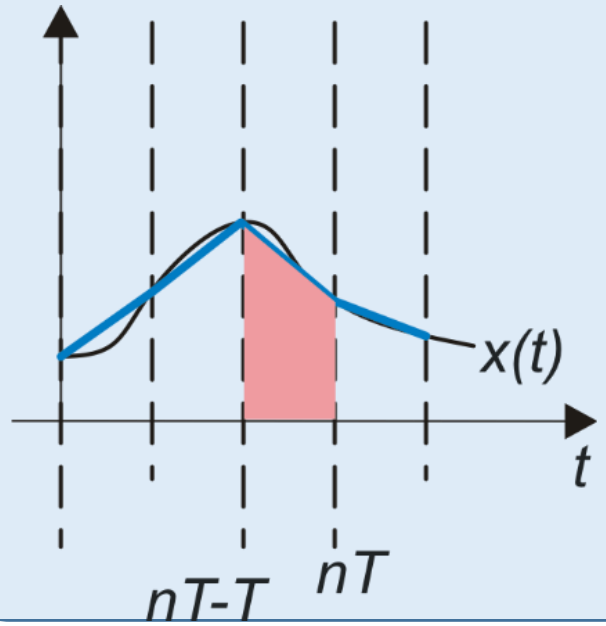
$T\omega_0 < 1$ Monotone step response

CT->DT transformation

$$H_I(z) = \frac{T}{z-1} \leftrightarrow \frac{1}{s} \quad \Rightarrow \quad \frac{z-1}{T} \to s$$

Simple integral substitution corresponds to state variable simulation

Trapezoidal integration



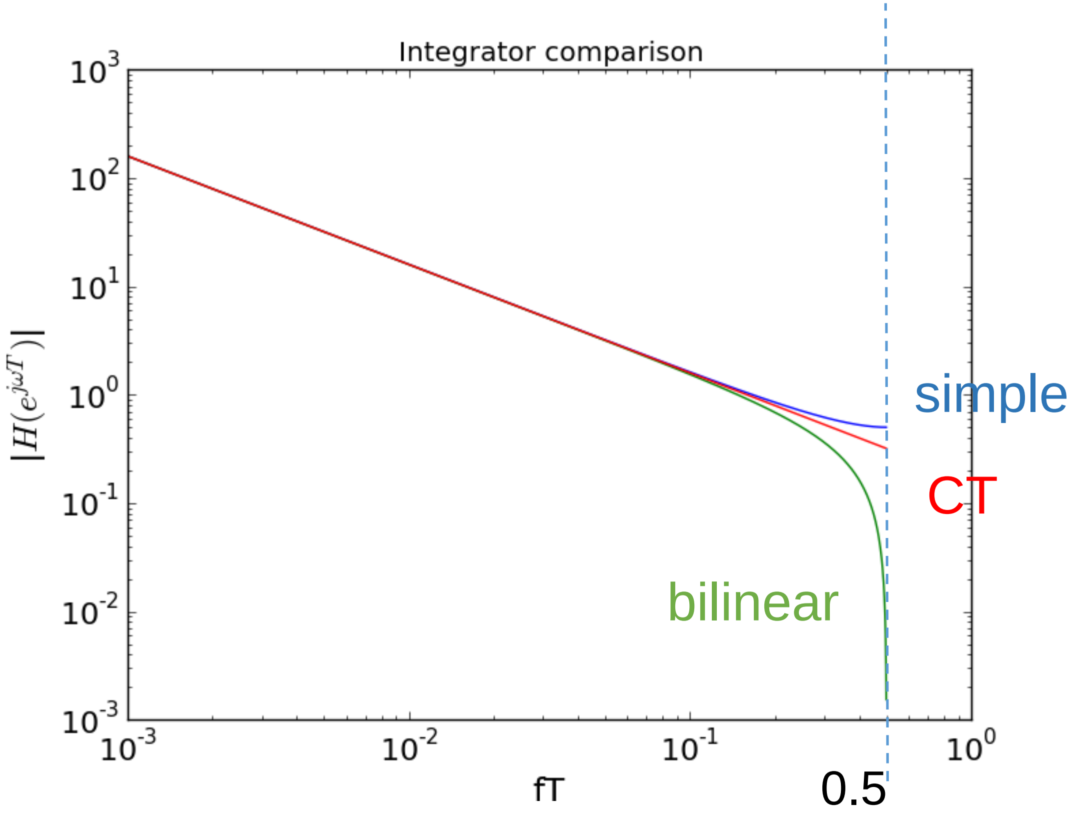
Bilinear Transformation

$$y(n) = y(n-1) + T\left[\frac{x(n)+x(n-1)}{2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{T}{2}\frac{z+1}{z-1} \leftrightarrow \frac{1}{s}$$

$$\boxed{\frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \to s}$$

Bilinear transform: characteristics

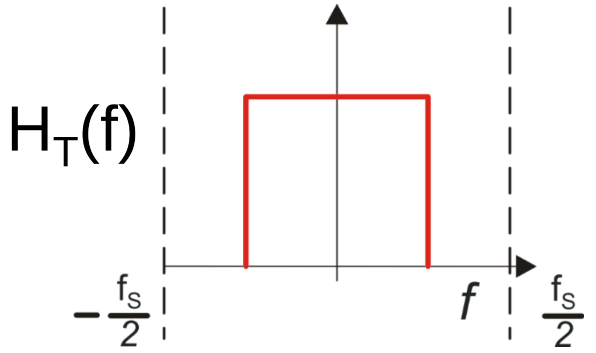


- Maintains stability
- all $s=j\omega$ are mapped to z belonging to the unit circle
- "Features" of the CT frequency response (e.g. peaks, notches are preserved
- Pre-warping of the CT singularities is necessary for close matching CT ⇔ DT

Design the CT filter with modified characteristic frequencies

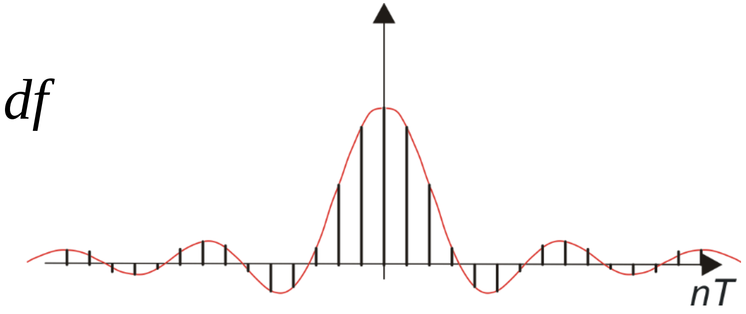
$$\frac{2}{T}\tan\left(\frac{\omega_i T}{2}\right) \rightarrow \omega_i$$

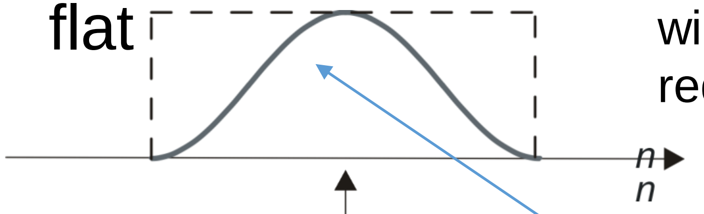
DT filter design from the impulse response



target filter (zero phase)

$$h(n) = T\int_{-f_S/2}^{f_S/2} H_T(f)e^{j2\pi fTn}df$$

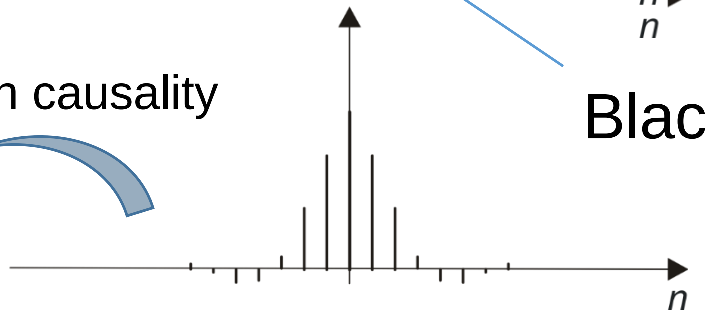




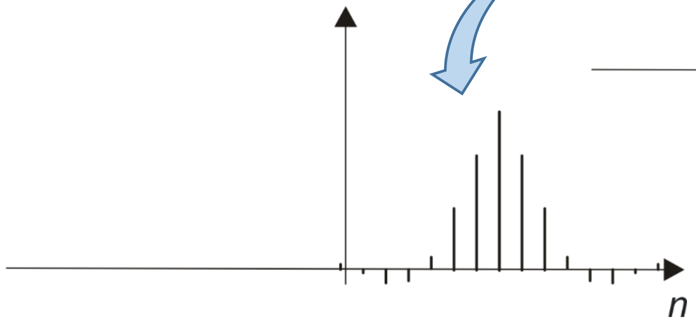
window
reduce samples to M+1

Blackman

Delay by M/2 to obtain causality



Windowed response
M = filter order

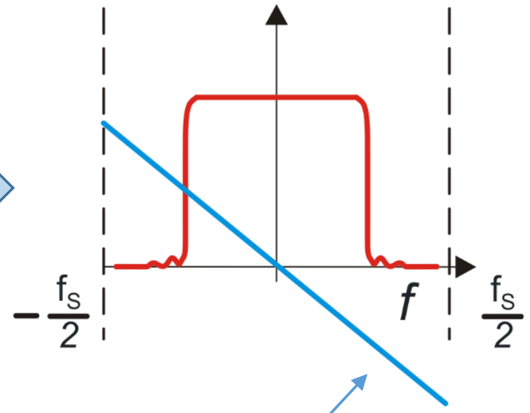


DT filter design from the impulse response

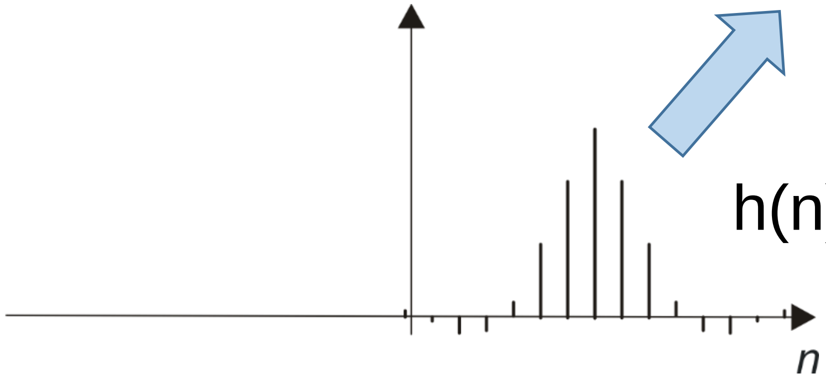
Obtained filter

Input signal

convolution



h(n)



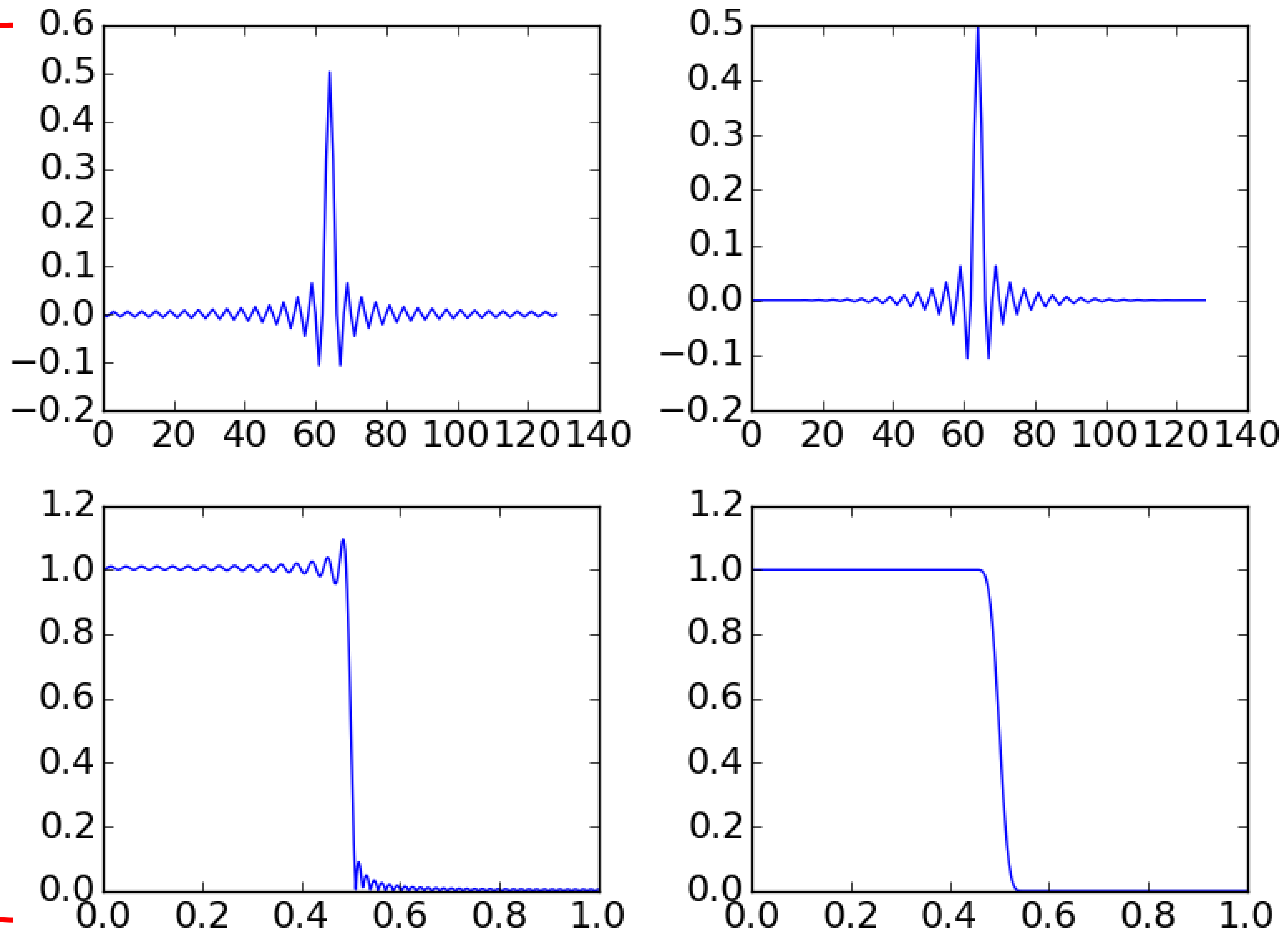
Linear phase delay
(MT/2 delay time)

Impulse response
(filter kernel)

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots + h(M)z^{-M}$$

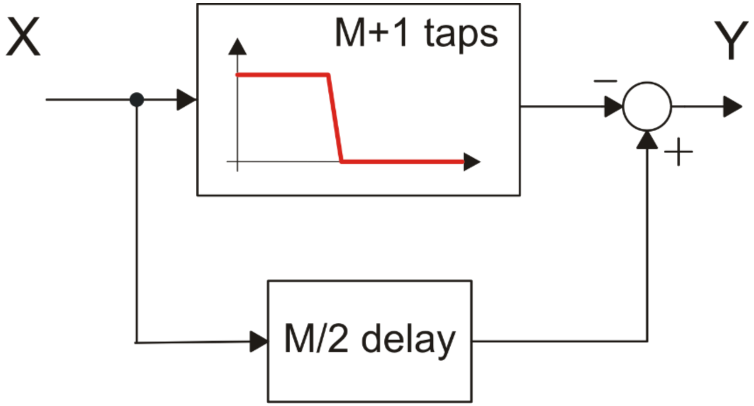
Effect of windowing

No window (flat)

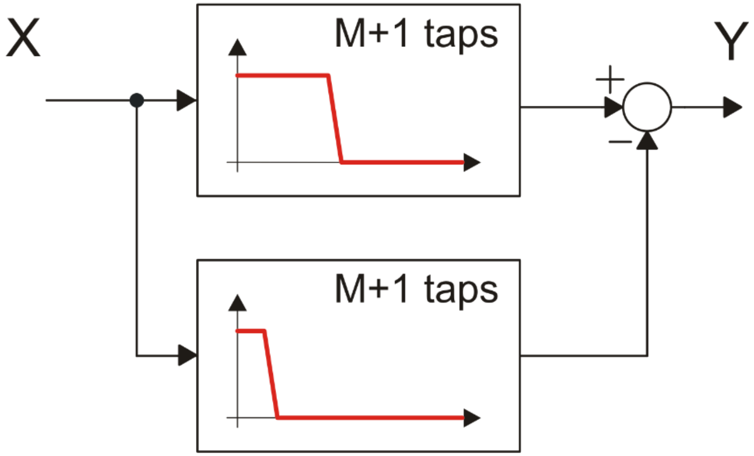


Blackman window

High pass and band-pass from low-pass



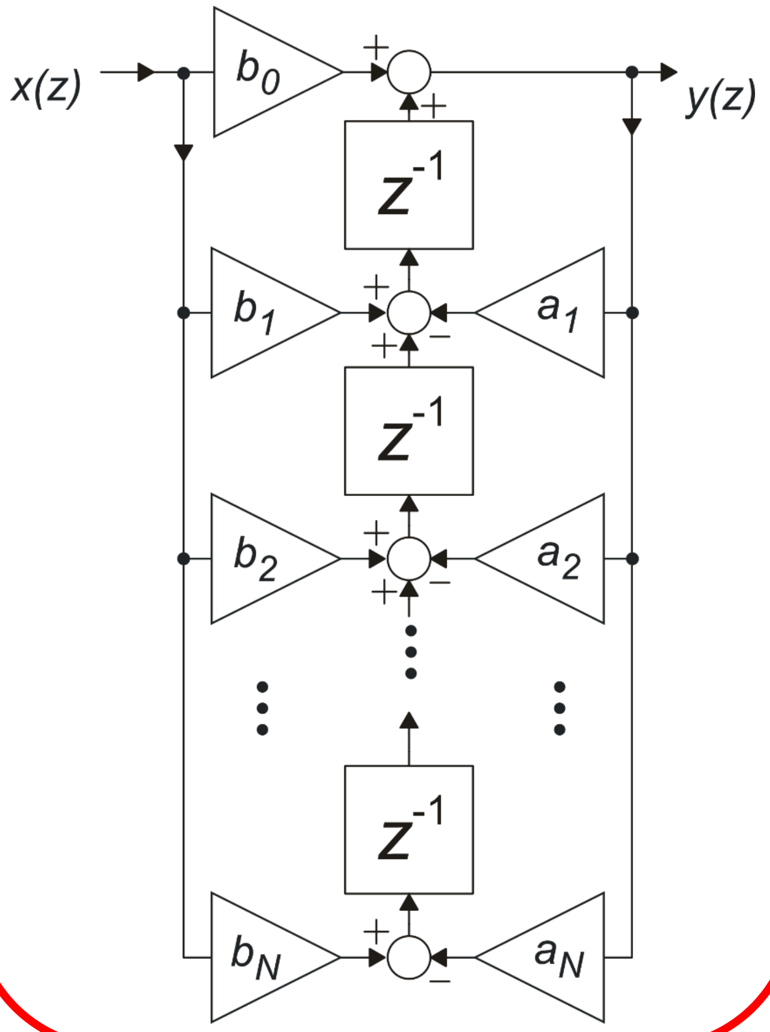
High pass



Band pass

Architecture of a generic DT filter

Direct structure



Coefficients b_i and a_i are called the "taps" of the filter and correspond to the coefficients of the numerator and denominator of H(z), according to:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots. + a_N z^{N}}$$

In a FIR filter the coefficients ai are all equal to zero, that is the denominator is = 1