Analog Filter Design

Part. 4: Discrete time filters

Sect. 4-a: General methods

Discrete time (DT) signals and sequences

- A discrete time signal is defined only on a “countable set” of time instants. The set, or time series, can be either finite or infinite.
- Definition of a discrete time signal can also disregard the actual times at which each value corresponds. In this way we have a pure, ordered sequence of values.

Discrete time signal: $x(t_n)$
Pure sequence: $x(n)$

- A discrete time signal **may** be the result of sampling a Continuous Time (CT)signal. Sampling is generally considered to be uniform. Generally, we are interested to DT signals for their capability to represent CT signals.

Discrete Time Signals (DTS): Linear operators

- As with CT signals, while dealing with DT signals we are interested in Linear, Time Invariant, Causal systems.
- In DT signals the derivative operator is substituted by the difference operator:

CT domain DT domain

$$\frac{dx(t)}{dt} \Leftrightarrow x(n) - x(n-1)$$

- More generally, the base operator in DT signal is the unity delay operator:

$$x(n) \Rightarrow x(n-1)$$

" T " operator

Difference Equations

- In the DT domain, differential equations are substituted by difference equations, where difference between elements of the sequences taken with different indexes (e.g. n, n-1, n-2 etc.) appears.

First (Δ) and second (Δ^2) difference definitions (non causal operators)

$$\Delta x(n) = x(n+1) - x(n)$$

$$\Delta^2 x(n) = \Delta x(n+1) - \Delta x(n) = x(n+2) - 2x(n+1) + x(n)$$

- Strictly speaking, difference equations are a particular case of recurrence equations:

$$y(n+1) = f\left[x(n+1), x(n),....x(n-k),\ \ y(n+1), y(n),....y(n-k)\right]$$

Analysis tool: Z transform

- In the case of linear, time invariant and causa recurrence equations, a powerful approach is using the Z-transform, which is the analogue of the Laplace transform.

- With the Z-transform, the unity delay operator is transformed into multiplication by Z^{-1} : recurrence equations becomes algebraic equations.

$$x(n) \quad \Rightarrow \quad x(n-1)$$

$$X(z) \quad \Rightarrow \quad z^{-1}X(z)$$

Common Z-Transform pairs

$$\delta(n) \leftrightarrow 1$$

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$u(n)a^n \leftrightarrow \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$u(n)na^n \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2} = \frac{az}{(z-a)^2}$$

Note that $u(n)a^n$ is an exponential function:

$$a^n = e^{\ln(a)n}$$

Exponential functions are eigenvectors of the delay operator

Z-Transform applied to LTI

LTI (Linear Time Invariant) system representation in the DT domain:

$$Y(z) = H(z)X(z) = \frac{N(z)}{D(z)}X(z)$$

Rational transfer function representations

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{N}}$$

Negative powers (preferred for synthesis)

$$H(z) = \frac{b_0 z^{M} + b_1 z^{M-1} + ... + b_{M-1} z^{M-1} + b_M}{a_0 z^{N} + a_1 z^{N-1} + ... + a_{N-1} z^{N-1} + ... + a_N} z^{N-M}$$

Positive powers

Z-Transform: a few properties

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X(z)$$ Final value theorem

$$x(0) = \lim_{z\to\infty} X(z)$$ Initial value theorem

DC gain of a transfer function *H(z)* $= \lim_{z\to 1} H(z)$

$$H(z) = \frac{N(z)}{D(z)}$$ ⇔ Stability: for all poles (D(z) roots) z_i:

$|z_i| \leq 1$

DT signals: Discrete-Time Fourier Transform (DTFT)

We consider a signal that is sampled with:
Sampling frequency $f_c = 1/T$, where T is the sampling interval

$$x(nT) = \int_{-f_c/2}^{+f_c/2} X_F(f) e^{j2\pi fnT} df$$

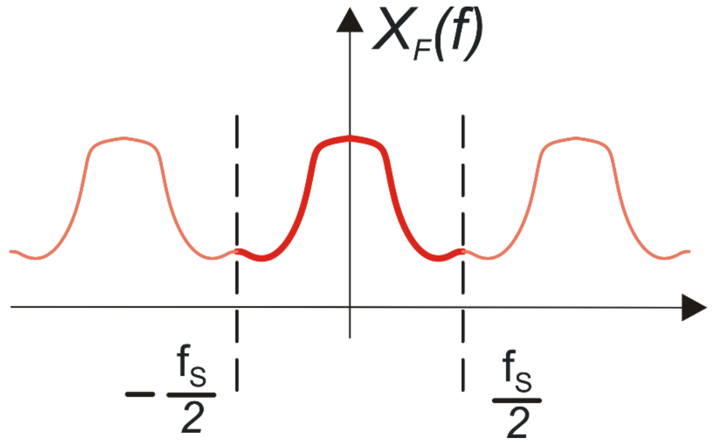
Decomposition of x(nT) into an integral of complex exponential functions

The frequecy domain is only fc wide since the exponential sequences are invariant for a frequency shift of kf_c

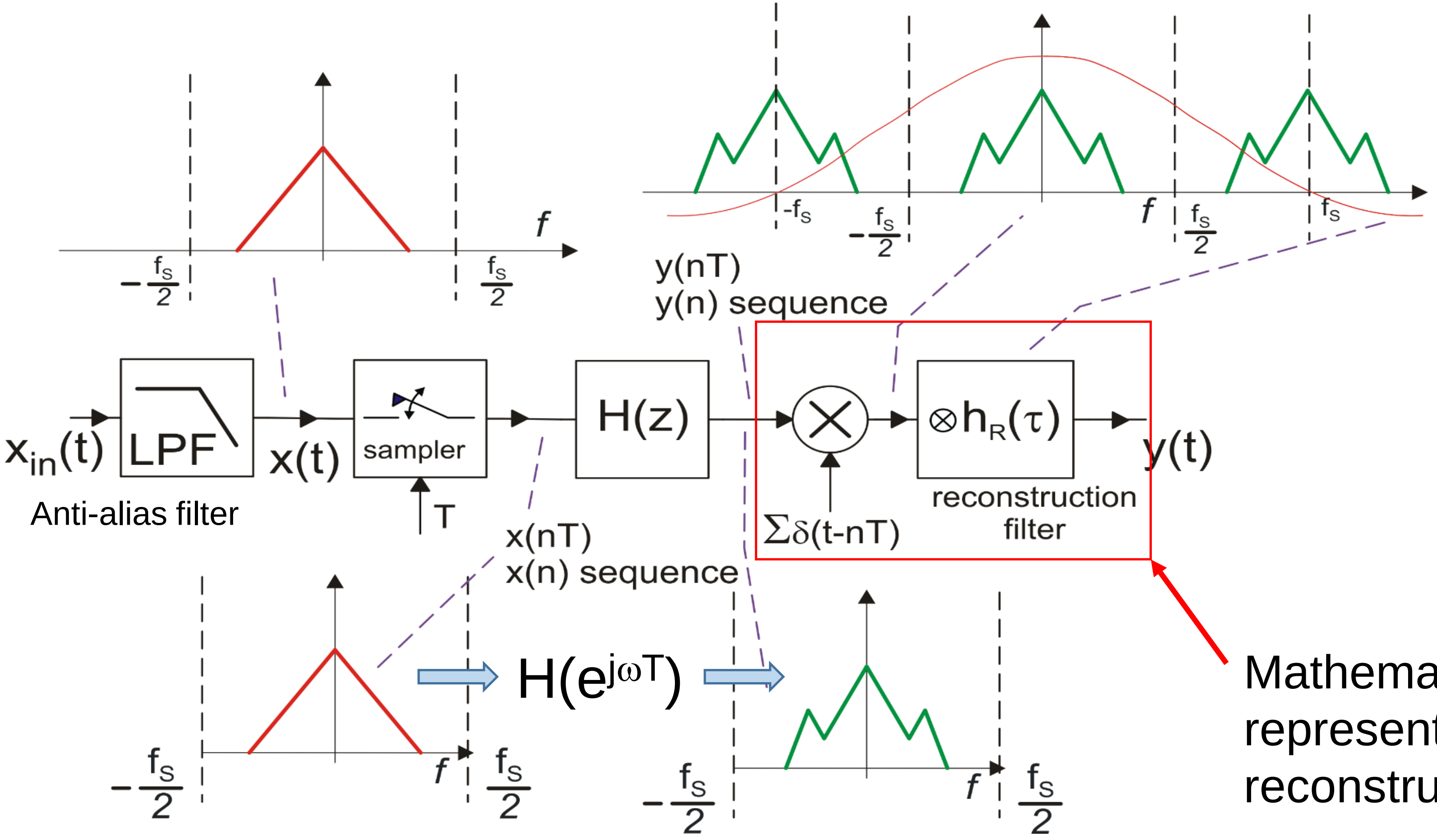
$$f \rightarrow f + kf_c \Rightarrow e^{j2\pi fTn} \rightarrow e^{j2\pi fTn} e^{j2\pi kf_c Tn} = e^{j2\pi fTn} e^{j2\pi kn} = e^{j2\pi fTn}$$

$$X_F(f) = T \sum_{n=-\infty}^{\infty} x(nT) e^{-j2\pi fnT}$$

With this definition the DTFT is periodical with period= f_c. Then, only the $[-f_c/2, f_c/2]$ domain is considered.



DT filters used to replace CT filters



Mathematical representation of the reconstruction process

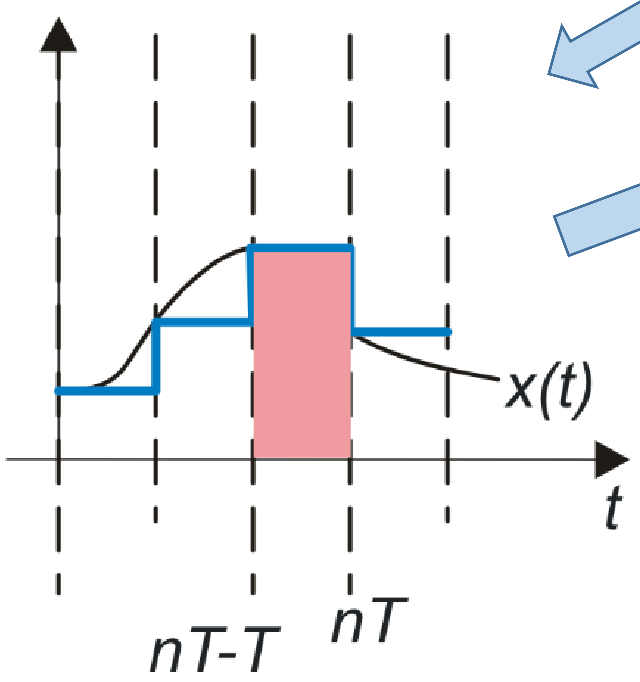
DT Filters synthesis (Ideal block diagrams)

IIR: Infinite Impulse Response *FIR: Finite Impulse Response*

- Start from a CT state variable filter and replace the CT integrator with DT integrator ⇨ IIR
- Start from a CT transfer function, $H_{CT}(s)$, and transform it into a DT transfer function H(z) ⇨ IIR
- Use synthesis approaches that do not need an analog filter as a starting point: use the delayed impulse response properly windowed ⇨ FIR

Method 1: simulate CT state variable filters

Integral approximation



$$y(t)=\int_0^t x(\tau)d\tau \Leftrightarrow \frac{x(s)}{s}$$

Time continuous

$$y(nT)=y(nT-T)+Tx(nT-T)$$

$$y(n)=y(n-1)+Tx(n-1)$$

$$Y(z)=z^{-1}Y(z)+Tz^{-1}X(z)$$

Forward Euler Integration

$$H_I(z)=T\frac{z^{-1}}{1-z^{-1}}=T\frac{1}{z-1}$$

Forward and Backward Euler Integrators

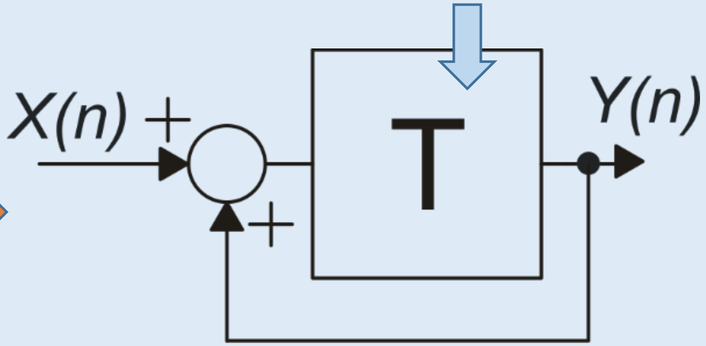
Forward

$$y(nT) = y(nT - T) + Tx(nT - T)$$

$$H_I(z) = T\frac{z^{-1}}{1 - z^{-1}} = T\frac{1}{z - 1}$$

1 cycle delay

$$\frac{z^{-1}}{1 - z^{-1}}$$

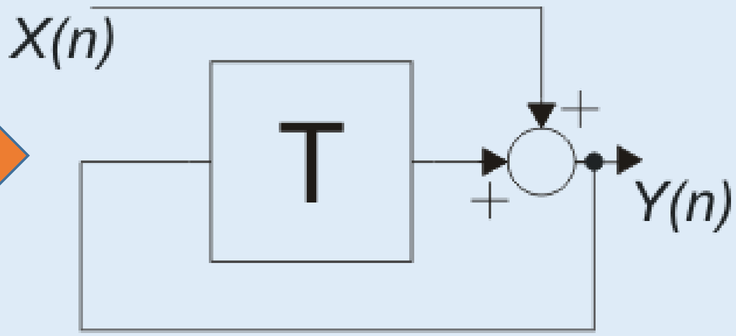


Backward

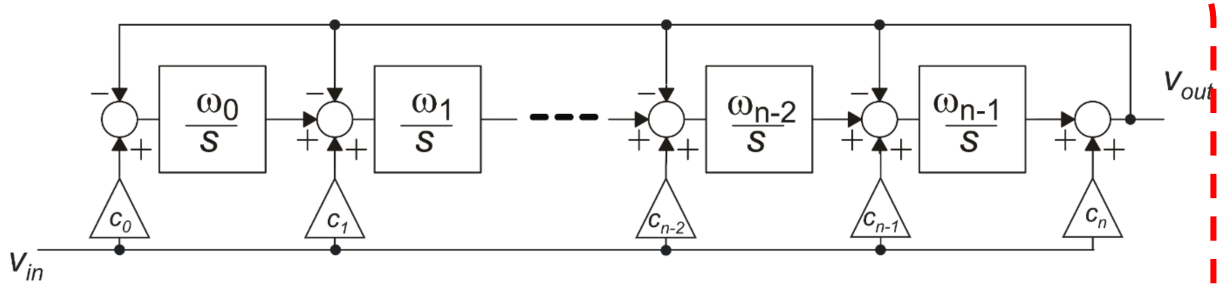
$$y(nT) = y(nT - T) + Tx(nT)$$

$$H_I(z) = T\frac{1}{1 - z^{-1}} = T\frac{z}{z - 1}$$

$$\frac{1}{1 - z^{-1}}$$



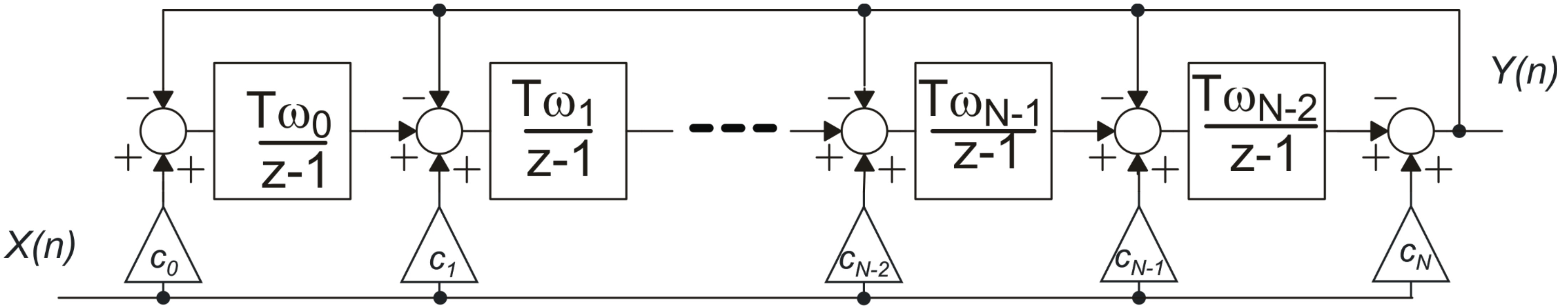
Simulation of CT state variable filters: result



Time continuous filter

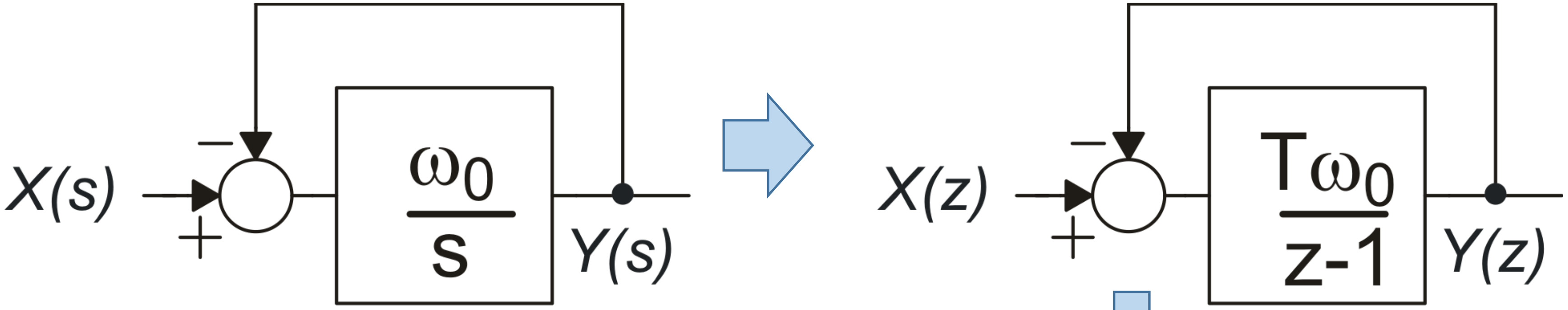
Example: use the Forward Euler Approx.

$$\frac{1}{s} \Rightarrow \frac{T}{z-1}$$



Discrete time approximation of the CT filter

Example: Transform a 1st order low pass filter



$$H_{CT}(s) = \frac{\omega_0}{s+\omega_0}$$

$$Y(z) = \frac{T\omega_0}{z-1}\left[X(z) - Y(z)\right]$$

$$H(z) = \frac{T\omega_0}{z-1+T\omega_0} = \frac{T\omega_0 z^{-1}}{1+\left(T\omega_0 - 1\right)z^{-1}}$$

$$z_p = 1 - T\omega_0$$

$0 < T\omega_0 < 2$ stability

$T\omega_0 < 1$ Monotone step response

Method 2: z -> s direct substitution

Euler Backward and Forward approximation can be used to find possible substitution formulas:

Forward

$$H_I(z) = \frac{T}{z-1} \leftrightarrow \frac{1}{s} \quad \Rightarrow \quad \frac{z-1}{T} \to s$$

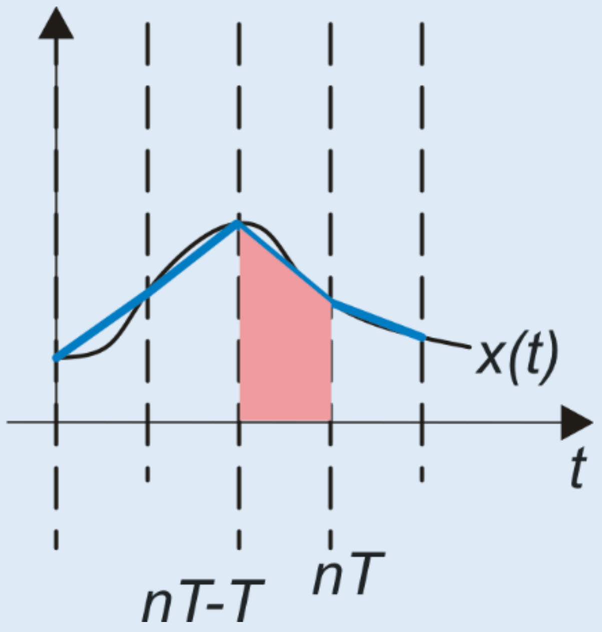
Backward

$$H_I(z) = \frac{zT}{z-1} \leftrightarrow \frac{1}{s} \quad \Rightarrow \quad \frac{z-1}{zT} \to s$$

A more precise formula for z-to-s substitution

The Bilinear Transformation

Trapezoidal integration



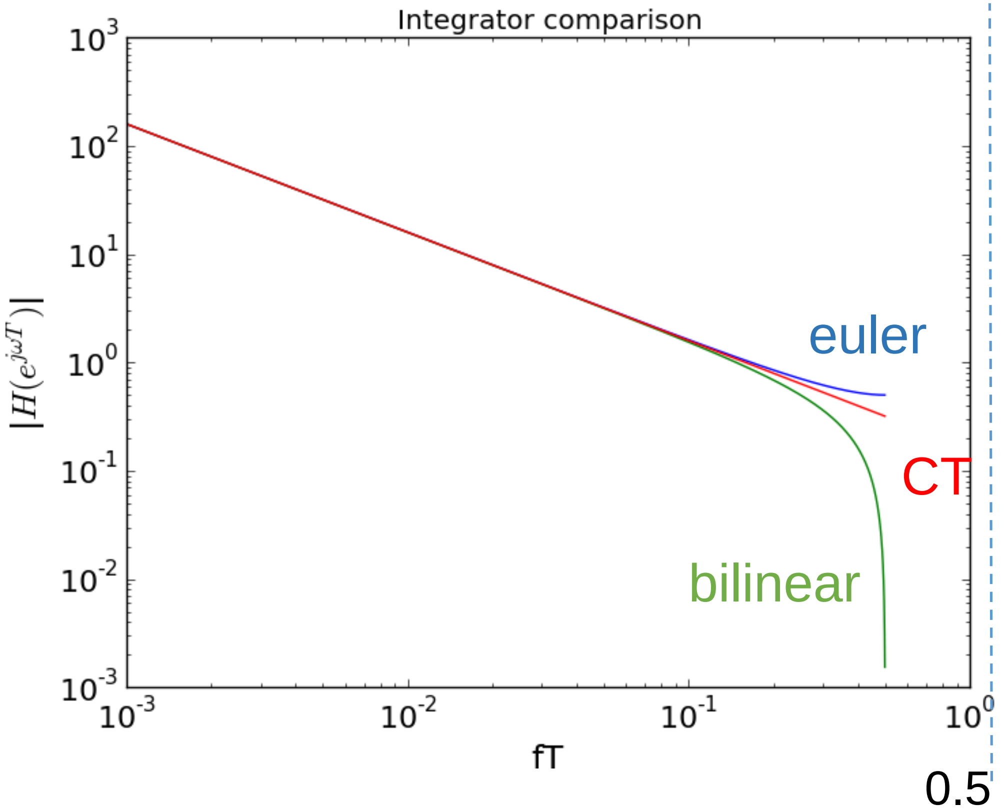
$$y(n)=y(n-1)+T\left[\frac{x(n)+x(n-1)}{2}\right]$$

$$\frac{Y(z)}{X(z)}=\frac{T}{2}\frac{z+1}{z-1}\leftrightarrow\frac{1}{s}$$

$$\frac{2}{T}\frac{z-1}{z+1}=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\rightarrow s$$

Bilinear transform: characteristics

Integrators compared

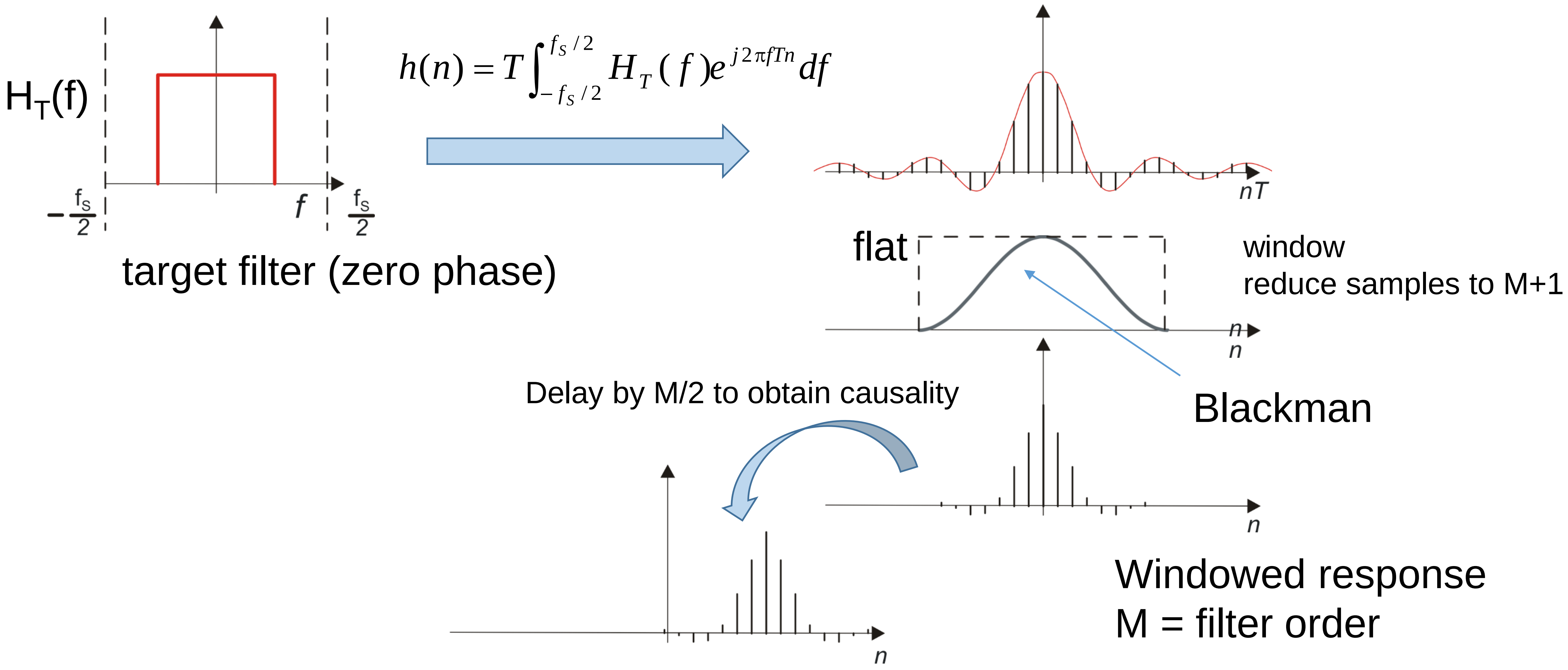


- Maintains stability
- all s=jω are mapped to z belonging to the unit circle
- “Features” of the CT frequency response (e.g. peaks, notches are preserved
- Pre-warping of the CT singularities is necessary for close matching CT ⇔ DT

Pre-warping

Design the CT filter with modified characteristic frequencies $\frac{2}{T}\tan\left(\frac{\omega_i T}{2}\right) \rightarrow \omega_i$

DT filter design from the impulse response

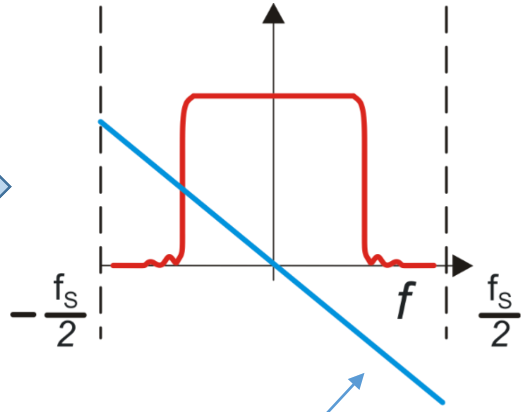


DT filter design from the impulse response

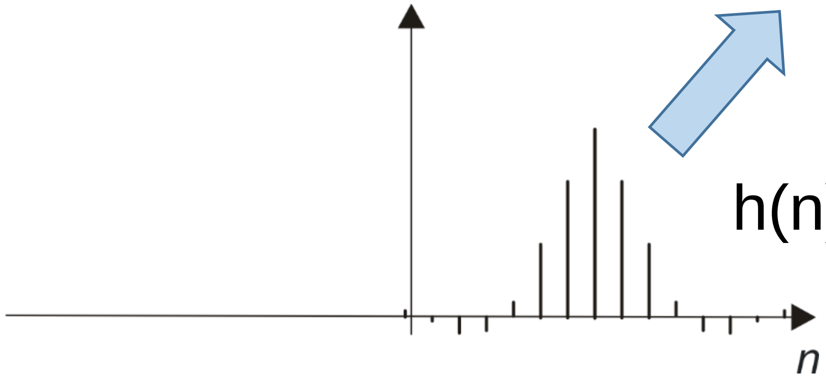
Input signal

convolution

Obtained filter



h(n)



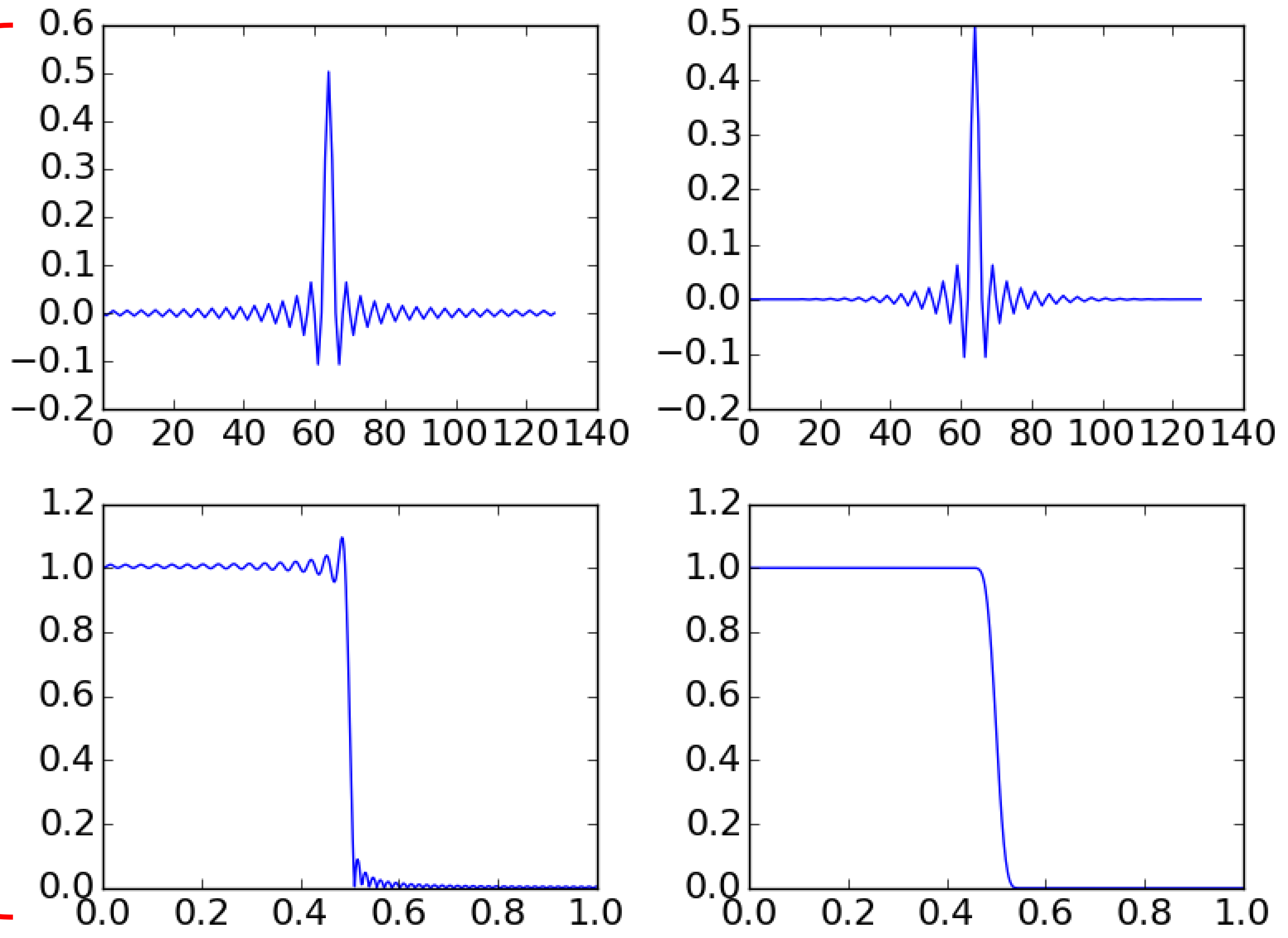
Impulse response
(filter kernel)

Linear phase delay
(MT/2 delay time)

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots + h(M)z^{-M}$$

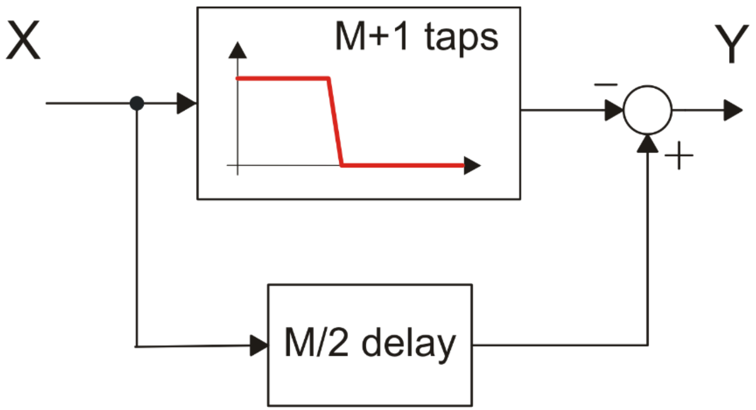
Effect of windowing

No window (flat)

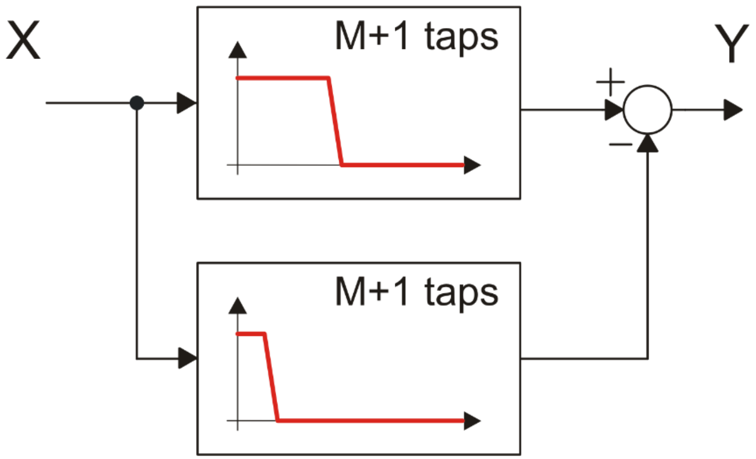


Blackman window

High pass and band-pass from low-pass



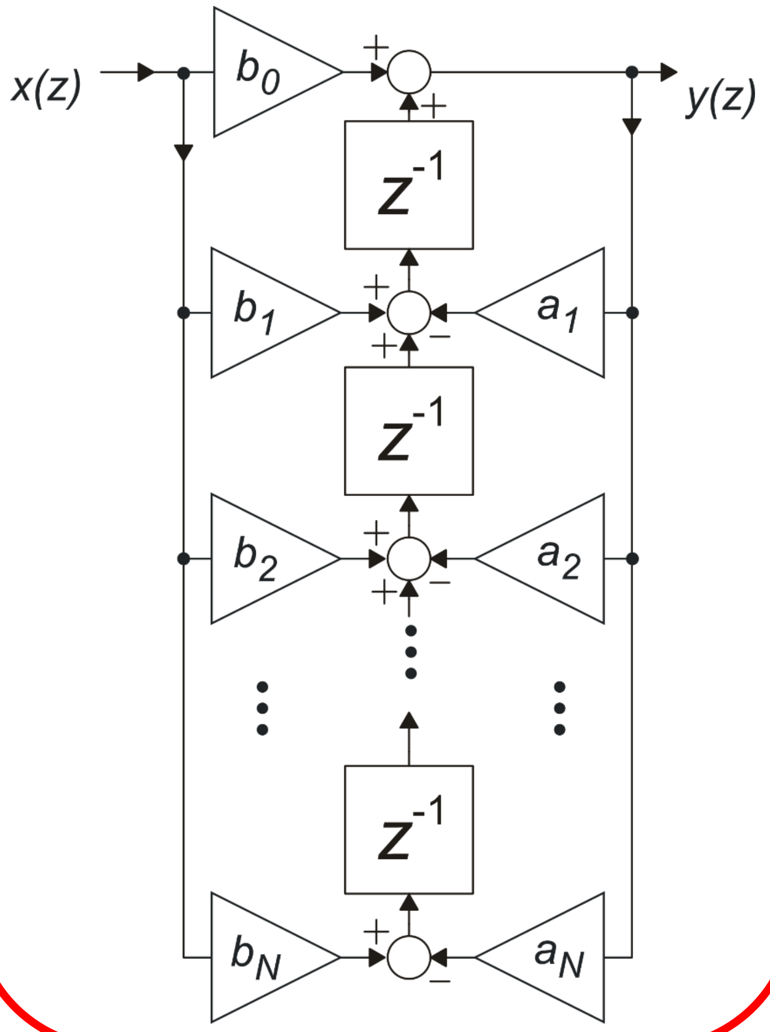
High pass



Band pass

Architecture of a generic DT filter

Direct structure



Coefficients b_i and a_i are called the "taps" of the filter and correspond to the coefficients of the numerator and denominator of H(z), according to:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots. + a_N z^{N}}$$

In a FIR filter the coefficients ai are all equal to zero, that is the denominator is = 1