3. LEO, MEO, HEO, what else?

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Kepler’s Laws

1. The planets move in a plane; the orbits described are ellipses with the Sun at one focus (1602).
2. The vector from the Sun to the planet sweeps equal areas in equal times (the law of areas, 1605).
3. The ratio of the square of the period $T$ of revolution of a planet around the sun to the cube of the semi-major axis $a$ of the ellipse is the same for all planets (1618).
1. The planet's satellite moves in a plane; the orbits described are ellipses with the sun Earth at one focus.
GPS Constellation

- 24 satellites
- (Almost) Circular Orbits on 6 equi-spaced planes
- Inclination of 55 degrees
- Altitude $h=20,200$ km
- Period (about) 12 hours (11h 58’ 2s)
- This 24-slot arrangement ensures users can view at least four satellites from virtually any point of the planet
Visibility of GPS Satellites

7 visible satellites
GALILEO Constellation

- 24 satellites (+6 spares)
- Circular Orbits on 3 equi-spaced planes
- Inclination of 56 degrees
- Altitude $h=23,222$ km
- Period (about) 14 hours
- Better coverage of high-latitudes than GPS
3. LEO, MEO, HEO, what else?

Second Kepler Law

- A
- Sun
- t
Second Kepler Law

3. LEO, MEO, HEO, what else?
• The vector from the sun to the planet sweeps equal areas in equal times (the law of areas, 1605).

\[ \Delta A = r \cdot \left( \frac{r \cdot \omega \Delta t}{2} \right) \Rightarrow \frac{dA}{dt} = \frac{\omega r^2}{2} = \text{constant} \]

• So another formulation is: at any point

\[ \omega r^2 = \text{constant} \]
- **Russian Satellites to cover very high-latitude areas**
- **The satellite stays at apogee for a long time (2\textsuperscript{nd} law...)**
Coverage of Molniya

Apogee

± 3 hours from apogee

± 4 hours from apogee
**Parameters of the Orbit**

- $a, b$ semi-major and semi-minor ellipse axes
- $c = (a^2 - b^2)^{1/2}$
- $r_a, r_p$ altitude of apogee/perigee
- $r_p = a - c$, $r_a = a + c$
- $e = c/a$ eccentricity

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}$$

$$= \frac{r_a - r_p}{r_a + r_p}$$
### Table 1. Orbital Data for the Planets

<table>
<thead>
<tr>
<th>Planet</th>
<th>Semimajor Axis (AU)*</th>
<th>Period (y)</th>
<th>Eccentricity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>0.39</td>
<td>0.24</td>
<td>0.21</td>
</tr>
<tr>
<td>Venus</td>
<td>0.72</td>
<td>0.6</td>
<td>0.01</td>
</tr>
<tr>
<td>Earth</td>
<td>1</td>
<td>1.00</td>
<td>0.02</td>
</tr>
<tr>
<td>Mars</td>
<td>1.52</td>
<td>1.88</td>
<td>0.09</td>
</tr>
<tr>
<td>(Ceres)</td>
<td>2.77</td>
<td>4.6</td>
<td>0.08</td>
</tr>
<tr>
<td>Jupiter</td>
<td>5.20</td>
<td>11.86</td>
<td>0.05</td>
</tr>
<tr>
<td>Saturn</td>
<td>9.54</td>
<td>29.46</td>
<td>0.06</td>
</tr>
<tr>
<td>Uranus</td>
<td>19.19</td>
<td>84.01</td>
<td>0.05</td>
</tr>
<tr>
<td>Neptune</td>
<td>30.06</td>
<td>164.82</td>
<td>0.01</td>
</tr>
</tbody>
</table>

*1 au = 149,597,870.707 km
Parameters of the Orbit

For any point on the elliptical orbit

\[ r = a \frac{1 - e^2}{1 + e \cos(v)} \]

And the speed of the satellite is

\[ v = \sqrt{GM \left(\frac{2}{r} - \frac{1}{a}\right)} \]

\[ GM = \mu \text{ standard gravitational parameter } 3.986 \times 10^{14} \text{ m}^3/\text{s}^2 \]
Computation of the Satellite Speed 1/2

- Conservation of energy (kinetic+gravitational potential)

\[
\frac{mv^2}{2} + \left( -\frac{GMm}{r} \right) = \text{constant}
\]

- @ perigee/apogee, + conservation of angular momentum

\[
\frac{v_a^2}{2} + \left( -\frac{GM}{r_a} \right) = \frac{v_p^2}{2} + \left( -\frac{GM}{r_p} \right) \quad \Rightarrow \quad mv_a r_a = mv_p r_p
\]

- So

\[
\frac{v_a^2}{2} \left( 1 - \frac{r_a^2}{r_p^2} \right) = GM \left( \frac{1}{r_a} - \frac{1}{r_p} \right) \quad \Rightarrow \quad \frac{v_a^2}{2} = GM \frac{r_p}{r_a} \frac{1}{r_p + r_a}
\]
But

\[ r_p + r_a = 2a \quad \Rightarrow \quad \frac{v^2_a}{2} = GM \frac{r_p}{r_a} \frac{1}{2a} = GM \frac{2a - r_a}{2a \cdot r_a} = GM \left( \frac{1}{r_a} - \frac{1}{2a} \right) \]

so we can generalize and get back to the start: for any point

\[ \frac{v^2}{2} = GM \left( \frac{1}{r} - \frac{1}{2a} \right) \quad \Rightarrow \quad v = \sqrt{GM \left( \frac{2}{r} - \frac{1}{a} \right)} \]

For circular orbits \((r=a)\)

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>Radius (km)</th>
<th>Period (s)</th>
<th>Velocity (m s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>6578</td>
<td>5309</td>
<td>7784</td>
</tr>
<tr>
<td>290</td>
<td>6668</td>
<td>5419</td>
<td>7732</td>
</tr>
<tr>
<td>800</td>
<td>7178</td>
<td>6052</td>
<td>7450</td>
</tr>
<tr>
<td>20000</td>
<td>26378</td>
<td>42636</td>
<td>3887</td>
</tr>
<tr>
<td>35786</td>
<td>42164</td>
<td>86164</td>
<td>3075</td>
</tr>
</tbody>
</table>
Megaconstellations: StarLink

About 3,000 satellites as of Sept. 2022, out of a planned total of FCC-approved 12,000

Nominal altitude: 550 km

Inclination: 53 degrees
Megaconstellations: OneWeb

Planned total of 684 small satellites

Nominal altitude: 1200 km

Inclination: 86.4 degrees (quasi-polar)
As for the Third Law...

- The ratio of the square of the period $T$ of revolution of a planet around the sun to the cube of the semi-major axis $a$ of the ellipse is the same for all planets (1618).

$$T^2 = \alpha \cdot a^3$$

- For the GEO we found

$$G \frac{mM}{r^2} = m\omega^2 r \implies \frac{GM}{a^3} = \frac{4\pi^2}{T^2}$$

- Therefore, we can find $\alpha$ from the GEO special case:

$$\alpha = \frac{4\pi^2}{GM} \implies T = 2\pi \sqrt{\frac{a^3}{GM}}$$
$$T = 2\pi \sqrt{\frac{(R + h)^3}{GM}}$$

$$v = \sqrt{\frac{GM}{(R + h)}}$$
As for the Third Law...

\[ T = 2\pi \sqrt{\frac{(R + h)^3}{GM}} \]

\[ v = \sqrt{\frac{GM}{(R + h)}} \]
As for the Third Law...

\[ T = 2\pi \sqrt{\frac{(R + h)^3}{GM}} \Rightarrow \omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{(R + h)^3}} \Rightarrow v = \omega (R + h) = \sqrt{\frac{GM}{(R + h)}} \]
• **Coordinates of the Earth Station (terminal)**
  – Left-handed Cartesian coordinate system $x,y,z$ with $O$ at the center of the Earth, the $x$-$y$ plane coincident with the equatorial plane, and the $x$-axes lying in the polar plane containing Greenwich
  – Spherical coordinates system $r,\lambda,\phi$ with $O$ at the center of the Earth, the *longitude* angle measured wrt the polar plane containing Greenwich, the *latitude* angle $\lambda$ measured wrt the equatorial plane. Often, the *radius* $r$ is decomposed as $r=R+h$ with $h$ the *altitude* and $R$ the Earth radius
  – Same for the satellite, when needed

• **Coordinates of the Satellite wrt the Earth Station (ES)**
  – Define the tangential plane to the Earth surface containing the ES, two “local” $x$-$y$ axes lying on such plane called North and East, the North axes pointing the North pole, and a “local” $z$ axis called Up to make a left-handed system.
  – The “local longitude” is called *azimuth* $\alpha$ and the “local latitude” is called *elevation* $\nu$ (sometimes $\theta$). The “local radius” is of course the *altitude* $h$ as above
The Mother-of-All SatCom Orbits

- Satellite SAT latitude $0$, longitude $\lambda_S$
- Earth Terminal ET latitude $\phi$, longitude $\lambda_T$
- We need to compute the ET-SAT distance $d$
Computation of $d$

\[ d = d_e^2 + R_v^2 \]

\[ R_v = R \sin \phi \]

\[ R_e = R \cos \phi \]

\[ d_e^2 = (R + h)^2 + R_e^2 - 2 R_e (R + h) \cos (\sigma - \sigma_t) \]
\[ d^2 = (R + h)^2 + R^2 \cos^2(\phi) - 2(R + h)R \cos(\phi) \cos(\lambda_s - \lambda_t) + R^2 \sin^2(\phi) \]

- and finally

\[ d = \sqrt{(R + h)^2 + R^2 - 2(R + h)R \cos(\phi) \cos(\lambda_s - \lambda_t)} \]
To aim your dish: Elevation and Azimuth

- **Input:** your latitude $\lambda_T$ and longitude $\phi$, and the SAT longitude $\lambda_S$
- **Output:** the elevation $\nu$ and azimuth $\alpha$ to mount your parabolic antenna

**IT IS COMPLICATED**

**Elevation:**

$$\nu = \arccos\left(\frac{R + h}{d}\right) \sqrt{1 - \cos(\phi)^2 \cos^2(\lambda_s - \lambda_T)}$$

**Pseudo-Azimuth:**

$$\alpha_0 = \arcsin\left(\frac{|\sin(\lambda_s - \lambda_T)|}{\sqrt{1 - \cos(\phi)^2 \cos^2(\lambda_s - \lambda_T)}}\right)$$
\[ \nu = \arccos \left( \frac{R + h}{d} \right) \sqrt{1 - \cos^2(\phi) \cos^2 \left( \lambda_s - \lambda_T \right)} \]
Azimuth Chart (Northern Hemisphere, looking EAST)

\[\alpha = 180 - \alpha_0 \quad \text{(deg)}\]

Diagram showing the relationship between azimuth angle \(\alpha\) and terminal latitude \(\phi\). Curves represent different latitude differences \(\lambda_S - \lambda_T\).
\[ \alpha = 180 + \alpha_0 \quad \text{(deg)} \]
• **Come ricevere SKY a Pisa, Via Caruso (43.72N – 10.38E)**

– In Italia è possibile ricevere gratuitamente i canali satellitari della piattaforma Tivùsat che replica l’offerta del digitale terrestre (Rai Uno, Rai Due, Rai Tre, Canale 5, Italia 1, Rete 4, La 7), più altri numerosi canali anche in HD e 4K.
– Tivùsat trasmette dai satelliti HotBird della flotta Eutelsat (13° Est).
– Anche i canali di Sky vengono diffusi attraverso HotBird 13° Est, e per questo motivo, sul territorio nazionale, la stragrande maggioranza delle parabole sono orientate a sud verso questo satellite.
– In Italia è anche possibile ricevere i canali trasmessi dal satellite Astra 19,2° Est, che presenta una ricca offerta di televisioni europee e mondiali, con oltre 350 canali gratuiti (tra cui Eurosport).
3. LEO, MEO, HEO, what else?

Elevation Chart for Italy

@DII for Hot Bird $\lambda_S - \lambda_T = 13 - 10.38 = 2.62$ deg
@DII, $\phi = 43.72$ deg
Azimuth Chart for Italy, looking EAST

@DII for Hot Bird \( \lambda_s - \lambda_T = 13 - 10.38 = 2.62 \) deg

@DII, \( \phi = 43.72 \) deg, \( \alpha = 180 - \alpha_0 \) deg
How do they launch it?