17. GNSS, does it work? Even with smartphones?

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Range Measurement & Positioning

\[ r_i = (x_i, y_i, z_i) \quad \text{ECEF coordinates of satellite } #i \]

\[ r = (x_u, y_u, z_u) \quad \text{ECEF coordinate of the receiver} \]

\[ r_i = \text{measured range from satellite } #i \text{ to receiver} \]

\[
\begin{align*}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} &= r_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} &= r_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} &= r_3
\end{align*}
\]

Three unknowns, three (independent) equations

DONE!

You try, and you find you’re in outer space.
Measuring range: the clock bias

Satellite time (Clock)

12:00:00,000

Transmitted signal

(Unknown) Clock Bias: 500 ms

\[ \tau \]

Received signal

User Receiver Time (Clock)

12:00:00,500

12:00:00,500 + \tau

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17. GNSS, does it work? Even with smartphones?)
• What is actually measured is a pseudorange $\rho$, containing the unknown clock bias effect

• Fortunately, the bias is the same for all measurement – can be considered as a fourth unknown to be found

• WE NEED ONE MORE SATELLITE/OBSERVATION/EQUATION

• Minimum # of satellites in view (received) for GNNS to work: 4
The (Nonlinear) Positioning Equations

\[
\begin{align*}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t &= \rho_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t &= \rho_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t &= \rho_3 \\
\sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t &= \rho_4
\end{align*}
\]

\[
\mathbf{\rho} \triangleq (\rho_1, \rho_2, \rho_3, \rho_4)^T, \quad \mathbf{\xi} \triangleq (x_u, y_u, z_u, c \cdot \Delta t)^T
\]

\[
f_i(\mathbf{\xi}) \triangleq \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} + c \cdot \Delta t
\]

\[
\mathbf{f}(\mathbf{\xi}) \triangleq (f_1(\mathbf{\xi}), f_2(\mathbf{\xi}), f_3(\mathbf{\xi}), f_4(\mathbf{\xi}))^T
\]

\[
\mathbf{\rho} = \mathbf{f}(\mathbf{\xi})
\]
The shape of the Earth

- **The Earth is Flat**
  - In early Egyptian and Mesopotamian thought, the world was portrayed as a disk floating in the ocean.

- **The Earth is a Sphere**
  - The founder of scientific geodesy was Eratosthenes (276-195 BC) of Alexandria who, assuming the Earth was spherical, deduced from measurements a radius for the Earth.

- **The Earth is an Ellipsoid**
  - Towards the end of the 17th century, Newton demonstrated that the concept of a truly spherical Earth was inadequate as an explanation of the equilibrium of the ocean surface, owing to the Earth rotation: he showed, by means of a simple theoretical model, that the hydrostatic equilibrium would be maintained if the equatorial axis were longer than the polar axis. This is equivalent to the statement that the body is flattened towards the pole.

- **The Earth is the Geoid**
  - Listing (1873) had given the name geoid to the “equipotential surface of the Earth’s gravity field which would coincide with the ocean surface, if the Earth were undisturbed and without topography.”
“Geodesy is the science concerned with the study of the shape and size of the Earth in the geometric sense as well as with the form of the equipotential surfaces of the gravity potential”
Earth-Centered Earth-Fixed (ECEF) coordinates

**Origin:** Earth’s center of mass

**Z-Axis:** direction of mean rotational axis of Earth

**X-Axis:** intersection of Greenwich meridian and the plane passing through the origin and normal to the Z-Axis

**Y-Axis:** direction orthogonal to Z-Axis and X-Axis

The reference system for our positioning equations

\[
\begin{align*}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t &= \rho_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t &= \rho_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t &= \rho_3 \\
\sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t &= \rho_4
\end{align*}
\]
The Earth is assumed to be spherical

**distance:**

\[ r = \sqrt{x_{ECEF}^2 + y_{ECEF}^2 + z_{ECEF}^2} \]

**latitude:**

\[ \varphi_{geoc} = \arctan\left( \frac{z_{ECEF}}{\sqrt{x_{ECEF}^2 + y_{ECEF}^2}} \right) \]

**longitude:**

\[ \lambda_{geoc} = \arctan\left( \frac{y_{ECEF}}{x_{ECEF}} \right) \]

**altitude:**

\[ h_{geoc} = r - r_E \]
The most used and very accurate reference frame is the World Geodetic System–1984 (WGS-84) Reference Ellipsoid

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-major axis</td>
<td>$a$</td>
<td>6378137 m</td>
</tr>
<tr>
<td>Eccentricity</td>
<td>$e_e$</td>
<td>$\sqrt{\frac{a^2 - b^2}{a^2}}$</td>
</tr>
<tr>
<td>Flattening</td>
<td>$e_p$</td>
<td>$\frac{a - b}{a}$ (\frac{1}{298.277223563})</td>
</tr>
</tbody>
</table>
Assuming the Earth as an ellipsoid:

**latitude:**

\[
\varphi_{\text{geod}} = \arctan \left( \frac{z_{\text{ECEF}} \cdot \frac{2e^2_p - e^2_p}{1 - e^2_p} \cdot a \cdot \sin^3 \theta}{p \cdot (2e^2_p - e^2_p) \cdot a \cdot \cos^3 \theta} \right)
\]

**longitude:**

\[
\lambda_{\text{geod}} = \lambda_{\text{geoc}} = \arctan \left( \frac{y_{\text{ECEF}}}{x_{\text{ECEF}}} \right)
\]

**altitude:**

\[
h_{\text{geod}} = \frac{p}{\cos \varphi_{\text{geod}}} - \nu
\]

where:

\[
p = \sqrt{x_{\text{ECEF}}^2 + y_{\text{ECEF}}^2}, \quad \theta = \arctan \left( \frac{z_{\text{ECEF}}}{p \cdot (1 - e^2_p)} \right), \quad \nu = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_{\text{geod}}}}
\]
User-centric reference that we use to locate a Satellite in the sky or a celestial body with respect to the observer position

**Origin:** observer’s position

**u-Axis:** direction of local vertical

**n-Axis:** direction of the North pole

**e-Axis:** direction orthogonal to u-Axis and n-Axis

**ENU**

*(East–North–Up)* coordinates
Topocentric reference system (2/2)

\[ \Delta \xi = \{ \Delta x, \Delta y, \Delta z \} = \xi_C - \xi_O, \quad \xi = \{ x_{ECEF}, y_{ECEF}, z_{ECEF} \} \]

\[
\begin{pmatrix}
E \\
N \\
U
\end{pmatrix} =
\begin{pmatrix}
-\sin \lambda_{\text{geod}} & \cos \lambda_{\text{geod}} & 0 \\
-\sin \varphi_{\text{geod}} \cos \lambda_{\text{geod}} & -\sin \varphi_{\text{geod}} \sin \lambda_{\text{geod}} & \cos \varphi_{\text{geod}} \\
\cos \varphi_{\text{geod}} \cos \lambda_{\text{geod}} & \cos \varphi_{\text{geod}} \sin \lambda_{\text{geod}} & \sin \varphi_{\text{geod}}
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{pmatrix}
\]

**elevation:** \( \varepsilon = \arctan \left( U / \sqrt{E^2 + N^2} \right) \)

**azimuth:** \( \alpha = \arctan \left( E / N \right) \)

**range:** \( \rho = \sqrt{E^2 + N^2 + U^2} \)
The (Nonlinear) Positioning Equations

\[ \begin{align*}
\sqrt{(x_u - x_1)^2 + (y_u - y_1)^2 + (z_u - z_1)^2} + c\Delta t &= \rho_1 \\
\sqrt{(x_u - x_2)^2 + (y_u - y_2)^2 + (z_u - z_2)^2} + c\Delta t &= \rho_2 \\
\sqrt{(x_u - x_3)^2 + (y_u - y_3)^2 + (z_u - z_3)^2} + c\Delta t &= \rho_3 \\
\sqrt{(x_u - x_4)^2 + (y_u - y_4)^2 + (z_u - z_4)^2} + c\Delta t &= \rho_4
\end{align*} \]

\[ f_1(\xi) \triangleq \sqrt{(x_u - x_i)^2 + (y_u - y_i)^2 + (z_u - z_i)^2} + c \cdot \Delta t \]

\[ f(\xi) \triangleq (f_1(\xi), f_2(\xi), f_3(\xi), f_4(\xi))^T \]

\[ f(\xi) = \rho \]
Iterative Solution (Linearization)

- Nonlinear Positioning Equation: 
  \[ f(\xi) - \rho = 0 \]

- Iterative solution (Newton): 
  \[ \xi^{(n+1)} = \xi^{(n)} - \left( Jf(\xi^{(n)}) \right)^{-1} \left( f(\xi^{(n)}) - \rho \right) \]

\[ Jf(\xi) = \begin{pmatrix} 
\frac{x_u - x_1}{r_1} & \frac{y_u - y_1}{r_1} & \frac{z_u - z_1}{r_1} \\
\frac{x_u - x_2}{r_2} & \frac{y_u - y_2}{r_2} & \frac{z_u - z_2}{r_2} \\
\frac{x_u - x_3}{r_3} & \frac{y_u - y_3}{r_3} & \frac{z_u - z_3}{r_3} \\
\frac{x_u - x_4}{r_4} & \frac{y_u - y_4}{r_4} & \frac{z_u - z_4}{r_4} 
\end{pmatrix} \]

- The Jacobian matrix is updated at any iteration
- We need to accurately know the positions of the anchors (the satellites in the sky)
- GNSS receivers typically provide solutions every second – there’s a time of 1s to perfect iterations
Computing the user velocity – Doppler shift measurement

- It is usually derived from observation of carrier frequency Doppler shift on satellite # $i$ – deriving velocity as $\dot{\mathbf{r}}$ is no good because of error propagation

$$f_i = \left(1 - \frac{\mathbf{r}_i}{||\mathbf{r}_i||} \cdot \frac{\mathbf{v}_i}{c}\right) f_0 = f_0 + \Delta f_i$$

- What matters is the radial component of the satellite velocity
- If the user receiver is moving itself with velocity $\mathbf{v}_u$ (the unknown that we wish to estimate)

$$f_i = \left(1 - \frac{\mathbf{r}_i}{||\mathbf{r}_i||} \cdot \frac{\mathbf{v}_i - \mathbf{v}_u}{c}\right) f_0$$
Computing velocity and clock drift

- Assumption: the satellites speeds are known because they can be computed from the navigation message.

- The carrier frequencies of the satellites should all be equal to the nominal system $f_0$, they are slightly different because the different atomic clocks onboard the satellites may by slightly different – the corrections are sent down in the navigation message, so the actual individual transmitted frequency $f_{0,i}$ by satellite $i$ is known, so that the received frequency is

$$f_{R,i} = \left(1 - \frac{r_i}{\| r_i \|} \cdot \frac{(v_i - v_u)}{c}\right) f_{0,i}$$

- Further problem: the measured received frequency $f_i$ is inaccurate as it contains an (unknown) frequency bias $\delta$ of the local oscillator (common to all observations): $f_{R,i} = (1+\delta)f_i$ and we have a further unknown (as in positioning...)

$$(1+\delta)f_i = \left(1 - \alpha_i \cdot \frac{(v_i - v_u)}{c}\right)f_{0,i}$$

$$\alpha_i \triangleq \frac{r_i}{\| r_i \|}$$
Velocity Equations (4 satellites)

\[
\begin{align*}
\frac{cf_1}{f_{0,1}} + \frac{cf_1}{f_{0,1}} \delta &= c - \alpha_{x,1}(v_{x,1} - v_{x,u}) - \alpha_{y,1}(v_{y,1} - v_{y,u}) - \alpha_{z,1}(v_{z,1} - v_{z,u}) \\
\end{align*}
\]

- The unknowns are \(v_{x,u}, v_{y,u}, v_{z,u}\) and the clock offset \(\delta\), all of the other quantities are known – the equations are linear
- As a side effect, the clock-rate bias \(\delta\) is also derived
- Of course, previous derivation of position is needed to compute \(\alpha\)
Many error sources on the measurement of the pseudo range(s) (receiver noise, ionosphere, troposphere, multipath, etc.).

They are collected into an observation noise vector $\mathbf{w}$ that adds up to $\mathbf{p}$ but cannot be separated from it and “propagate” down to the solution $\mathbf{\xi}$.

The total noise variance, i.e., the sum of the variance on the 4 component is called the User Equivalent Range Error (UERE).

It is fundamental to understand the propagation of such errors become down to the (iterative) solution of the positioning equations.

New Model: $\mathbf{f}(\mathbf{\xi}) = \mathbf{p} + \mathbf{w}$

Linearized positioning equation: $\mathbf{f}(\mathbf{\xi}_0) + \mathbf{A}(\mathbf{\xi} - \mathbf{\xi}_0) = \mathbf{p} + \mathbf{w}, \mathbf{A} \triangleq \mathbf{Jf}(\mathbf{\xi}_0)$

Solving for $\mathbf{\xi}$: $\mathbf{\xi} = \mathbf{\xi}_0 + \mathbf{A}^{-1}(\mathbf{p} + \mathbf{w} - \mathbf{f}(\mathbf{\xi}_0)) \Rightarrow \Delta\mathbf{\xi} = -\mathbf{A}^{-1}\Delta\mathbf{f}(\mathbf{\xi}_0) + \mathbf{A}^{-1}\mathbf{w}$

Noise propagation equation: $\Delta\mathbf{\xi} = \mathbf{A}^{-1}\mathbf{w}$
Dilution of Precision

- We can find the covariance matrix $C_\Delta$ of the propagated noise

$$C_\Delta = E\left\{\Delta \xi \Delta \xi^T\right\} = E\left\{A^{-1}w w^T \left(A^T\right)^{-1}\right\} = A^{-1}E\left\{ww^T\right\}\left(A^T\right)^{-1}$$

- In a first approximation, we can assume that the components of $w$ are zero-mean and uncorrelated (this is not completely true for instance for the Iono term), so that

$$C_\Delta = A^{-1}C_w \left(A^T\right)^{-1} = A^{-1}\sigma_w^2 I \left(A^T\right)^{-1} = \sigma_w^2 \left(A^T A\right)^{-1}$$

- The equation describes the so-called *dilution of precision*, i.e., how the measurement inaccuracy propagates down to the positioning solution.

- An overall metrics of positioning error (including clock bias) is the total variance (the UERE), that is, the sum of the four variances of the four positioning components:

$$\sigma_{tot}^2 \triangleq \sigma_{\xi_1}^2 + \sigma_{\xi_2}^2 + \sigma_{\xi_3}^2 + \sigma_{\xi_4}^2 = \sigma_w^2 \text{tr}\left(A^T A\right)^{-1}$$
**Different DOPs**

**Vertical Dilution of Precision (VDOP)**

\[
VDOP \triangleq \frac{\sigma_{z_u}}{\sigma_w} = \sqrt{V_{33}} \quad \left( A^T A \right)^{-1} = \{ v_{ij} \}
\]

**Horizontal Dilution of Precision (HDOP)**

\[
HDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2}}{\sigma_w} = \sqrt{V_{11} + V_{22}}
\]

**Position Dilution of Precision (PDOP)**

\[
PDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2}}{\sigma_w} = \sqrt{V_{11} + V_{22} + V_{33}}
\]

**Time Dilution of Precision (TDOP)**

\[
TDOP \triangleq \frac{c \cdot \sigma_{\Delta t}}{\sigma_n} = \sqrt{V_{44}}
\]

**Geometrical Dilution of Precision (GDOP)**

\[
GDOP \triangleq \frac{\sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + c^2 \cdot \sigma_{\Delta t}^2}}{\sigma_w} = \sqrt{V_{11} + V_{22} + V_{33} + V_{44}}
\]
• The four Satellites are the vertices of a tetrahedron
• The larger is the volume of the tetrahedron, the smaller is the DOP
• What is needed is “diversity” across satellite positions: for instance, if the satellites tend to lie on a plane, the accuracy of the position is very bad
• The DOP values are of the order of the unity
• VDOP is larger than HDOP since all satellites are “on the same side” of the receiver vertically
More than 4 satellites in visibility: Least-Squares solution

- We can improve on the positioning accuracy using \( N > 4 \) pseudorange measurements coming from more than 4 satellites.
- We have \( N \) observed pseudoranges, and we can write \( N \) positioning equations; the Jacobian matrix \( \mathbf{A} \) is \( N \times 4 \), but we can use the very same iterative solution as before, replacing \( \mathbf{A} \) with the LS matrix

\[
\left( \mathbf{A}_{LS} \right)_{4 \times N} \triangleq \left( \mathbf{A}^T \mathbf{A} \right)^{-1} \mathbf{A}^T_{4 \times N}
\]

- And the new DOP matrix is

\[
\left( \mathbf{A}^T_{LS} \mathbf{A}_{LS} \right)^{-1}_{4 \times 4}
\]
### Typical GNSS Pseudorange Error Budget

<table>
<thead>
<tr>
<th>ERROR SOURCE</th>
<th>RMS ERROR (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbital (Ephemeris)</td>
<td>0.8</td>
</tr>
<tr>
<td>Satellite Clock</td>
<td>1</td>
</tr>
<tr>
<td>Receiver Noise</td>
<td>0.3</td>
</tr>
<tr>
<td>Ionospheric</td>
<td>7</td>
</tr>
<tr>
<td>Tropospheric</td>
<td>0.2</td>
</tr>
<tr>
<td>Multipath</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>7.2</strong></td>
</tr>
</tbody>
</table>
The Sagnac effect (1/2)

- The propagation speed of the electromagnetic wave generated by the satellite does not depend on the speed of the satellite (special relativity).
- During the propagation time of the satellite signal (about 67 ms), a clock on the surface of the Earth will experience a finite time-shift with respect to a resting (inertial) reference frame at the geocenter.
- The measured range is $r_S$ rather than $r$.
- It is also called “Earth Rotation Correction”
\[ r \approx \sqrt{r_S^2 - 2\Omega [x_s \cdot y_u - y_s \cdot x_u]} = \sqrt{r_S^2 - 2\omega_E \frac{r_s}{c} [x_s \cdot y_u - y_s \cdot x_u]} \]

\[ \omega_E = 7.292115 \times 10^{-5} \text{ rad/s (WGS-84)} \]
\[ c = 2.99792458 \times 10^8 \text{ m/s (WGS-84)} \]

\[ |r - r_s| \approx 3 \div 10 \text{ m for each satellite} \]
• Any GNSS receiver can “lock” to the satellites’ clock, since it derives its own clock bias $\Delta t$. Where does the satellites’ time come from? Who determines and keep it? What is its relation with time references available on the Internet?

• **Universal Time (UT1)**
  - Aka astronomical time or solar time, is determined by the position of the Sun relative to the observer. The exact duration of a UT1 day is not always the same - UT1 does not flow uniformly

• **International Atomic Time (TAI)**
  - Metrologic timescale maintained by the Bureau des Poids et Measure (BIPM)
  - TAI is defined as a coordinate timescale in a geocentric reference frame with the SI second as the scale unit realized on the rotating geoid

• **Coordinated Universal Time (UTC)**
  - stepped atomic time scale based on the rate of TAI adjusted by the addition or deletion of integer seconds, known as leap seconds, to maintain the time within 0:9 s of Universal Time (UT1),
### GNSS Time(s) vs. UTC

| UTC – GPST       | $0 - n + 19s + C_0$ | GPS Time (GPST) is steered to UTC(USNO), $C_0$ is required to be less than 1 μs but is typically less than 20 ns |
| UTC – GLST      | $-3h + 0s + C_1$   | GLONASSST (GLONASS Time) is steered to UTC(SU) including leap seconds. $C_1$ is required to be less than 1 ms. Note that GLONASSST is offset from UTC by −3 hours corresponding to the offset of Moscow local time from the Greenwich meridian. |
| UTC – GST       | $0 - n + 19s + C_2$ | Galileo Time (GST) is steered to a set of European Union UTC(k) realization and $C_2$ is nominally less than 50 ns. |
| UTC – BDT       | $0 - n + 33s + C_3$ | BeiDou Time (BDT) is steered to UTC(NTSC) and $C_3$ is specified to be maintained less than 100 ns. |

Each offset is separated into an integer number of seconds and its fractional (subsecond) component $C_i$.

In addition, $n = \text{TAI} – \text{UTC}$ denotes the integer second offset between International Atomic Time and Coordinated Universal Time (e.g., $n = 36$ s starting on 1 July 2015)