# COMPITO DELL' 11/01/2018 - FILA A

Es. 1 – Sia dato il seguente filtro analogico causale $H_a(s) = \dfrac{2}{(s+1)(s+3)}$.

1) Si calcoli la risposta in frequenza del suddetto filtro e se ne faccia il grafico di modulo e fase.

A partire dal filtro analogico si vuole progettare un filtro numerico con simili caratteristiche. A tal fine si utilizza l'inv. della risp. imp.

2) Supponendo di campionare con un tempo T sufficientemente elevato per ~~non~~ avere aliasing trascurabile, si scriva la funzione di trasferimento del nuovo filtro numerico e se ne individui la zona di conv.

3) Si scriva la espressione di $h(n)$, si calcoli il modulo della risposta in frequenza e, dopo averne calcolato il valore in $f=0$ e $f=\pm 1/2T$, se ne faccia il grafico.

4) Si applichi ora al filtro num. il segnale $x(n) = u(n)$. Si scriva la risposta $y(n)$ del filtro a tale segnale.

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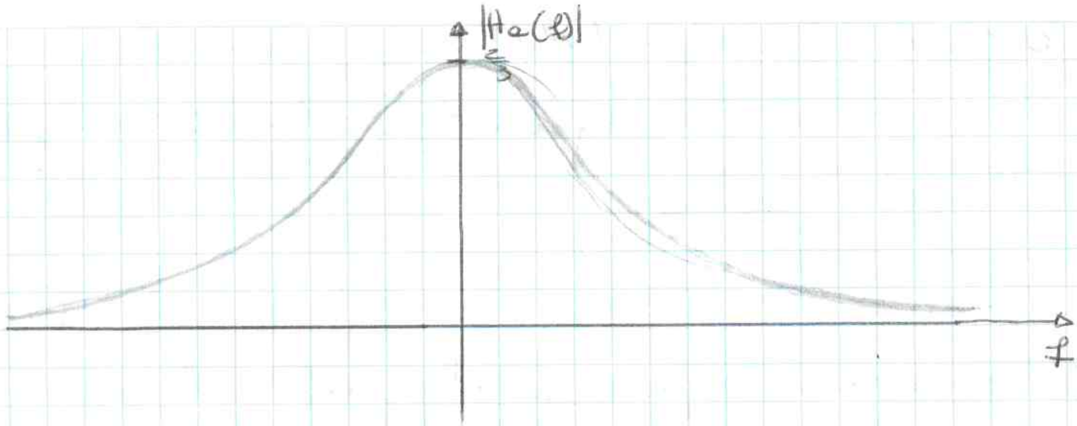
1) $$H_a(s) = \frac{2}{(s+1)(s+3)}$$

$$H_a(f) = \frac{2}{(j2\pi f + 1)(j2\pi f + 3)}$$

$$|H_a(f)| = \frac{2}{\sqrt{(1+4\pi^2 f^2)(9+4\pi^2 f^2)}}$$




2) Si può scrivere $H_a(s)$ come

$$H_a(s) = \frac{1}{s+1} - \frac{1}{s+3}$$

$$\Rightarrow h_a(t) = e^{-t}u(t) - e^{-3t}u(t)$$

$$h(n) = T h_a(nT) = T\left(e^{-nT} - e^{-3nT}\right) u(n)$$

$h(0) = 0$ Non ci sono problemi di discontinuità

$$H(z) = \frac{T}{1 - e^{-T}z^{-1}} - \frac{T}{1 - e^{-3T}z^{-1}}$$

$$= T\left(e^{-T} - e^{-3T}\right) \frac{z^{-1}}{\left(1 - e^{-T}z^{-1}\right)\left(1 - e^{-3T}z^{-1}\right)}$$

$$= T\left(e^{-T} - e^{-3T}\right) \frac{z}{\left(z - e^{-T}\right)\left(z - e^{-3T}\right)}$$

Il filtro ha 2 poli $z_{p1} = e^{-T}$ e $z_{p2} = e^{-3T}$
Essendo causale esso converge per $|z| > e^{-T}$

3) $\bar{H}(f) = H(z)\Big|_{z = e^{j2\pi fT}}$ poiché la circ. di raggio unitario è compreso nella zona di convergenza.

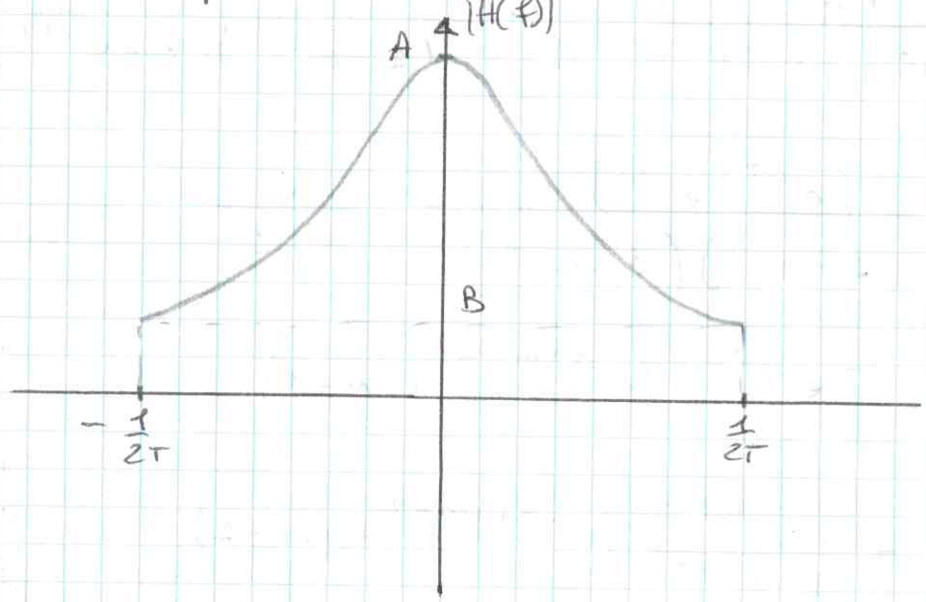
$$\bar{H}(f) = T\left(e^{-T}-e^{-3T}\right)\frac{e^{j2\pi fT}}{\left(e^{j2\pi fT}-e^{-T}\right)\left(e^{j2\pi fT}-e^{-3T}\right)}$$

$$\left|\bar{H}(f)\right| = \frac{T\left(e^{-T}-e^{-3T}\right)}{\sqrt{\left(1+e^{-2T}-2e^{-T}\cos 2\pi fT\right)\left(1+e^{-6T}-2e^{-3T}\cos 2\pi fT\right)}}$$

$$\left|\bar{H}(0)\right| = \frac{T\left(e^{-T}-e^{-3T}\right)}{\left(1-e^{-T}\right)\left(1-e^{-3T}\right)} = A$$

$$\left|\bar{H}\left(\pm\frac{1}{2T}\right)\right| = \frac{T\left(e^{-T}-e^{-3T}\right)}{\left(1+e^{-3T}\right)\left(1+e^{-3T}\right)} < \left|H(0)\right|$$

È un filtro passa-basso.




4) $x(n) = u(n) \Rightarrow X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

$$Y(z) = H(z)X(z) = \underbrace{T\left(e^{-T}-e^{-3T}\right)}_{K}\frac{z^2}{\left(z-e^{-T}\right)\left(z-e^{-3T}\right)(z-1)}$$

Sviluppiamo in fratti parziali

$$\frac{z^2}{\left(z-e^{-T}\right)\left(z-e^{-3T}\right)(z-1)}$$

$$\frac{z^2}{(\ )(\ )(\ )} = \frac{A_1}{(z-e^{-T})} + \frac{A_2}{(z-e^{-3T})} + \frac{A_3}{z-1}$$

$$A_1 = \left.\frac{z^2}{(z-e^{-3T})(z-1)}\right|_{z=e^{-T}} = \frac{e^{-2T}}{(e^{-T}-e^{-3T})(e^{-T}-1)}$$

$$A_2 = \left.\frac{z^2}{(z-e^{-T})(z-1)}\right|_{z=e^{-3T}} = \frac{e^{-6T}}{(e^{-3T}-e^{-T})(e^{-3T}-1)}$$

$$A_3 = \left.\frac{z^2}{(z-e^{-T})(z-e^{-3T})}\right|_{z=1} = \frac{1}{(1-e^{-T})(1-e^{-3T})}$$

Riassumendo

$$Y(z) = \frac{Te^{-2T}}{(e^{-T}-1)(z-e^{-T})} + \frac{Te^{-6T}}{(1-e^{-3T})(z-e^{-3T})}$$

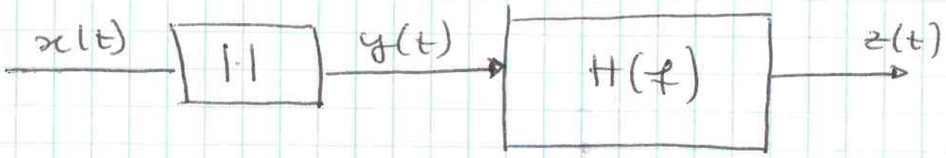
$$+ \frac{T(e^{-T}-e^{-3T})}{(1-e^{-T})(1-e^{-3T})} \cdot \frac{1}{z-1}$$

$$y(n) = \frac{Te^{-2T}}{e^{-T}-1}\left(e^{-T}\right)^{n-1} u(n-1)$$

$$+ \frac{Te^{-6T}}{1-e^{-3T}}\left(e^{-3T}\right)^{n-1} u(n-1)$$

$$+ \frac{T(e^{-T}-e^{-3T})}{(1-e^{-T})(1-e^{-3T})}\, u(n-1)$$

Es. 2 – Il segnale periodico $x(t) = \sum_{k=-\infty}^{+\infty} (-1)^k x_0\left(t - \frac{kT}{2}\right)$, con $x_0(t) = \frac{t}{T}\, \mathrm{rect}\left(\frac{2t}{T}\right)$, costituisce l'ingresso del sistema di figura 1 in cui la non-linearità iniziale è costituita da $y = |x|$ e $H(f) = \mathrm{tr}\left(\frac{fT}{16}\right)$.




1) Dopo aver fatto il grafico di $y(t)$, calcolarne la potenza e la Trasformata ~~[illegible]~~ di Fourier.

2) Fare il grafico di $Z(f)$ e scrivere l'espressione del segnale $z(t)$.

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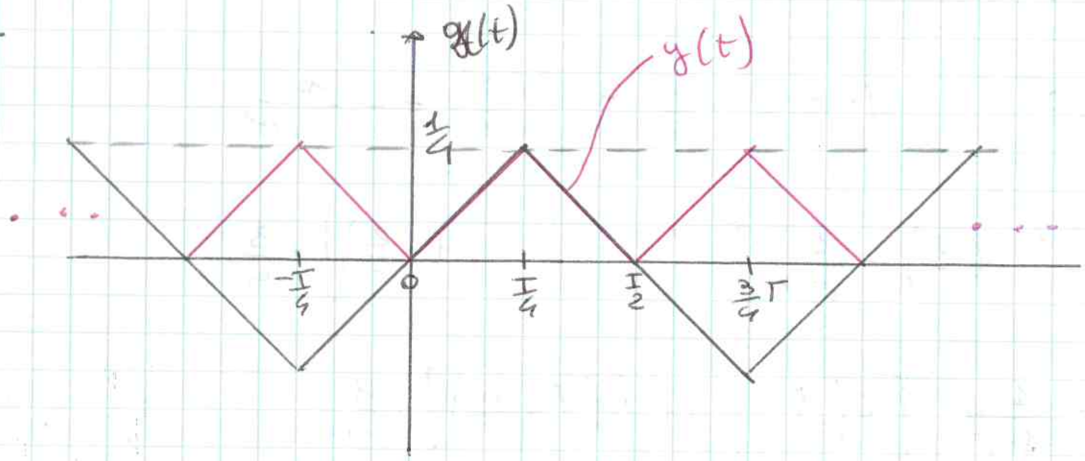
1)




Dalla figura si osserva che il periodo del segnale $y_0(t)$ è $T/2$, dunque

$$y(t) = \sum_{k=-\infty}^{+\infty} y_0\left(t - \frac{kT}{2}\right) \qquad \text{con} \qquad y_0(t) = \frac{1}{4}\, \mathrm{tr}\left(\frac{t - T/4}{T/2}\right)$$

$$Y_k = \frac{2}{T}\, Y_0\left(\frac{2k}{T}\right) \quad ; \quad Y_0(f) = \frac{T}{16}\, \mathrm{sinc}^2\left(\frac{fT}{4}\right) e^{-j\frac{\pi f T}{2}}$$

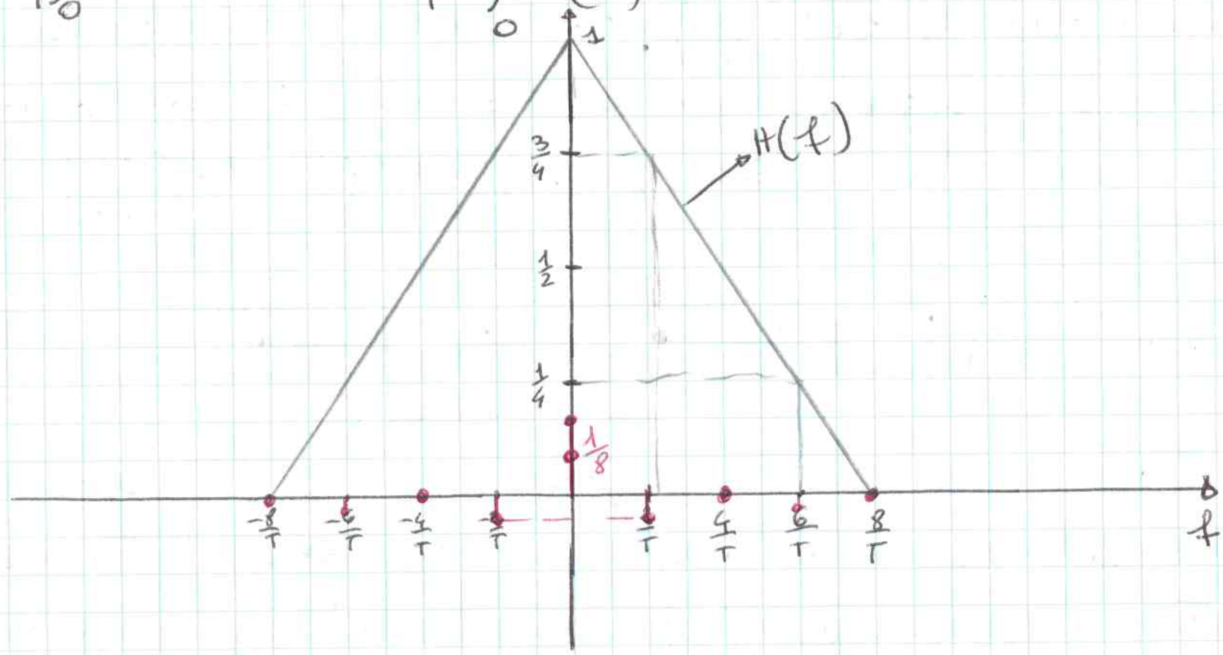
$$Y_k = \frac{1}{8}\, \mathrm{sinc}^2\left(\frac{k}{2}\right) e^{-jk\pi} = \frac{(-1)^k}{8}\, \mathrm{sinc}^2\left(\frac{k}{2}\right)$$

La Trasformata generalizzata è data da

$$\boxed{Y(f) = \frac{1}{8} \sum_{k=-\infty}^{+\infty} (-1)^k\, \mathrm{sinc}^2\left(\frac{k}{2}\right) \delta\left(f - \frac{2k}{T}\right)}$$

$$P_Y = \frac{2}{T}\int_0^{\frac{T}{2}} y_0^2(t)\,dt = \frac{4}{T}\int_0^{\frac{T}{4}} \left(\frac{t}{T}\right)^2 dt = \frac{1}{48}$$

2)




$$Y_0 = \frac{1}{8} \qquad Y_{\pm 1} = -\frac{1}{8}\,\mathrm{sinc}^2\left(\frac{1}{2}\right) = -\frac{1}{2\pi^2}$$

$$Y_{\pm 2} = 0 \qquad Y_{\pm 3} = -\frac{1}{8}\,\mathrm{sinc}^2\left(\frac{3}{2}\right) = -\frac{1}{18\pi^2}$$

$$Z(f) = \frac{1}{8}\delta(f) - \frac{1}{2\pi^2}\left[\delta\left(f-\frac{2}{T}\right) + \delta\left(f+\frac{2}{T}\right)\right]\cdot\frac{3}{4}$$

$$-\frac{1}{18\pi^2}\left[\delta\left(f-\frac{6}{T}\right) + \delta\left(f+\frac{6}{T}\right)\right]\cdot\frac{1}{4}$$

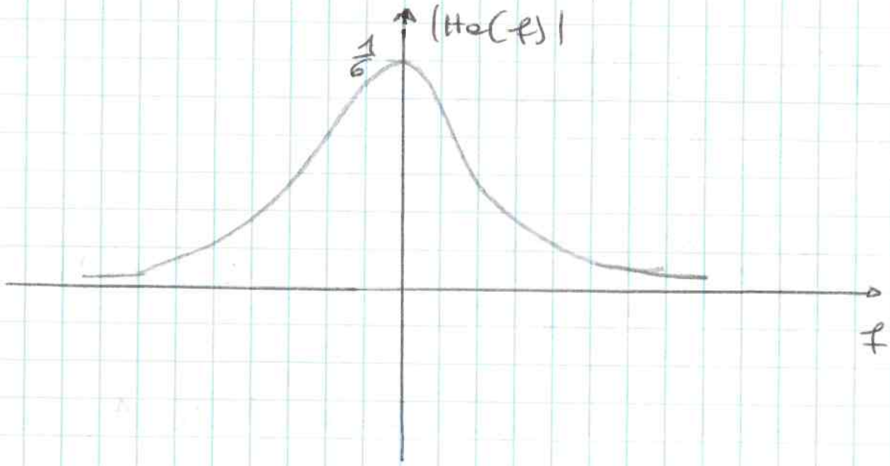
$$\Rightarrow z(t) = \frac{1}{8} - \frac{3}{4\pi^2}\cos\left(4\pi\frac{t}{T}\right) - \frac{1}{36\pi^2}\cos\left(\frac{12\pi t}{T}\right)$$

COMPITO DELL'11/01/2018 - FILA B

Simile a FILA A con $H_a(s) = \dfrac{1}{(s+2)(s+3)}$

1) $H_a(f) = \dfrac{1}{(j2\pi f+2)(j2\pi f+3)}$

$$|H_a(f)| = \frac{1}{\sqrt{(4+4\pi^2 f^2)(9+4\pi^2 f^2)}}$$




2) $H_a(s) = \dfrac{1}{s+2} - \dfrac{1}{s+3}$

$$h_a(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

$$h(n) = T h_a(nT) = T\left(e^{-2nT} - e^{-3nT}\right)u(n)$$

$h(0) = 0$ Non ci sono discontinuità

$$H(z) = \frac{T}{1-e^{-2T}z^{-1}} - \frac{T}{1-e^{-3T}z^{-1}}$$

$$= T\left(e^{-2T} - e^{-3T}\right)\frac{z^{-1}}{(1-e^{-2T}z^{-1})(1-e^{-3T}z^{-1})}$$

$$= T\left(e^{-2T} - e^{-3T}\right)\frac{z}{(z-e^{-2T})(z-e^{-3T})}$$

Converge per $|z| > e^{-2T}$

3) $\bar{H}(f) = H(z)\Big|_{z=e^{j2\pi fT}}$

$$\bar{H}(f) = \frac{T\left(e^{-2T} - e^{-3T}\right) \quad e^{j2\pi fT}}{\left(e^{j2\pi fT} - e^{-2T}\right)\left(e^{j2\pi fT} - e^{-3T}\right)}$$

$$|\bar{H}(f)| = \frac{T\left(e^{-2T} - e^{-3T}\right)}{\sqrt{\left(1 + e^{-4T} - 2e^{-2T}\cos 2\pi fT\right)\left(1 + e^{-6T} - 2e^{-3T}\cos 2\pi fT\right)}}$$

$$|\bar{H}(0)| = \frac{T\left(e^{-2T} - e^{-3T}\right)}{\left(1 - e^{-2T}\right)\left(1 - e^{-3T}\right)} = A$$

$$\left|\bar{H}\left(\pm\frac{1}{2T}\right)\right| = \frac{T\left(e^{-2T} - e^{-3T}\right)}{\left(1 + e^{-2T}\right)\left(1 + e^{-3T}\right)} = B < |H(0)|$$

Grafico simile a quello di File A.

4) $x(n) = 2^n u(n)$

$$X(z) = \frac{z}{z-2}$$

$$Y(z) = H(z)\, X(z) = T\underbrace{\left(e^{-2T} - e^{-3T}\right)}_{K} \frac{z^2}{\left(z - e^{-2T}\right)\left(z - e^{-3T}\right)\left(z-2\right)}$$

Sviluppiamo in fratti semplici

$$\frac{z^2}{(\ )(\ )(\ )} = \frac{A_1}{z - e^{-2T}} + \frac{A_2}{z - e^{-3T}} + \frac{A_3}{z-2}$$

$$A_1 = \frac{z^2}{\left(z - e^{-3T}\right)\left(z-2\right)}\Bigg|_{z=e^{-2T}} = \frac{e^{-4T}}{\left(e^{-2T} - e^{-3T}\right)\left(e^{-2T} - 2\right)}$$

$$A_2 = \left.\frac{z^2}{(z-e^{-2T})(z-2)}\right|_{z=e^{-3T}} = \frac{e^{-6T}}{(e^{-3T}-e^{-2T})(e^{-3T}-2)}$$

$$A_3 = \left.\frac{z^2}{(z-e^{-2T})(z-e^{-3T})}\right|_{z=2} = \frac{4}{(2-e^{-2T})(2-e^{-3T})}$$

Riassumendo

$$Y(z) = \frac{Te^{-4T}}{(e^{-2T}-2)(z-e^{-2T})} + \frac{Te^{-6T}}{(2-e^{-3T})(z-e^{-3T})}$$

$$+ \frac{4T(e^{-2T}-e^{-3T})}{(2-e^{-2T})(2-e^{-3T})} \frac{1}{z-2}$$

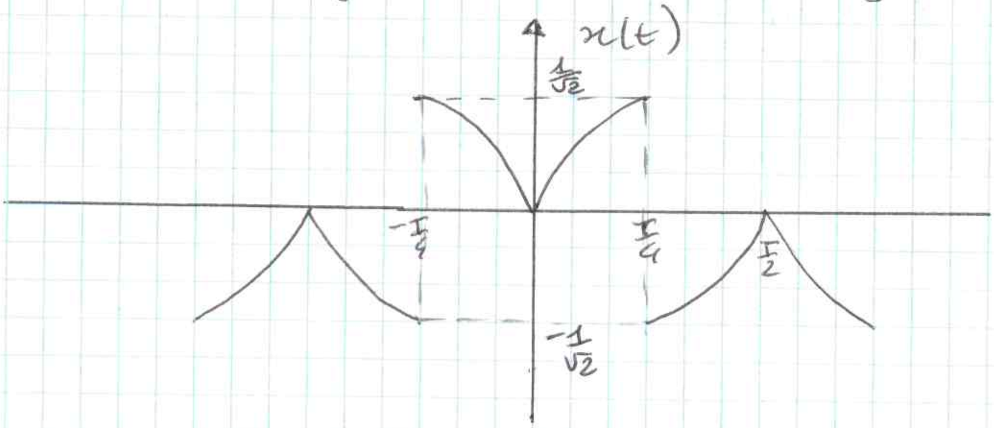
$$\Rightarrow y(n) = \frac{Te^{-4T}}{(e^{-2T}-2)}\left(e^{-2T}\right)^{n-1} u(n-1)$$

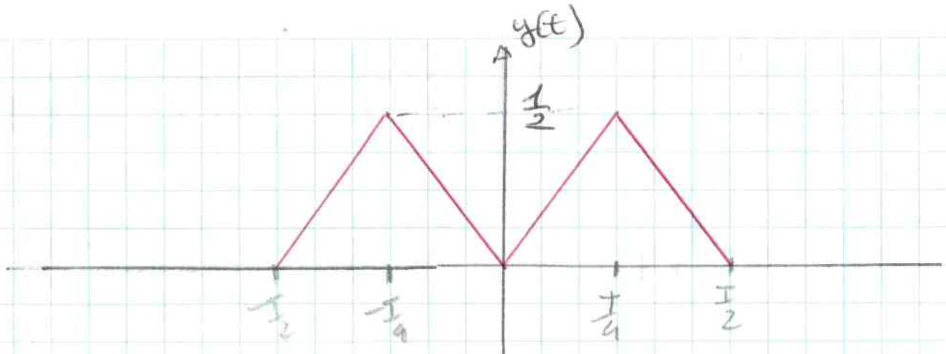
$$+ \frac{Te^{-6T}}{2-e^{-3T}}\left(e^{-3T}\right)^{n-1} u(n-1)$$

$$+ \frac{4T(e^{-2T}-e^{-3T})}{(2-e^{-2T})(2-e^{-3T})} 2^{n-1} u(n-1)$$

Es. 2 – Simile alla File A ma $x_0(t) = \sqrt{\frac{2|t|}{T}}\, \text{rect}\left(\frac{2t}{T}\right)$, $y = x^2$ e $H(f) = \text{rect}\left(\frac{fT}{10}\right)$.

Facciamo il grafico di $x(t)$ e $y(t)$.






Il periodo di $y(t)$ è $T/2$ e $y_0(t) = \frac{1}{2}\, tr\left(\frac{t - T/4}{T/2}\right)$

$$Y_0(f) = \frac{T}{8}\, \text{sinc}^2\left(\frac{fT}{4}\right) e^{-i\frac{\pi f T}{2}}$$

$$Y_k = \frac{(-1)^k}{4}\, \text{sinc}^2\left(\frac{k}{2}\right)$$

$$Y(f) = \frac{1}{4} \sum_{k=-\infty}^{+\infty} (-1)^k\, \text{sinc}^2\left(\frac{k}{2}\right) \delta\left(f - \frac{2k}{T}\right)$$

$P_Y =$

2) $Y_0 = \frac{1}{4}$ $\qquad Y_{\pm 1} = -\frac{1}{4}\, \text{sinc}^2\left(\frac{1}{2}\right) = -\frac{1}{\pi^2}$

$Y_{\pm 2} = 0$ $\qquad Y_{\pm 3} = -\frac{1}{4}\, \text{sinc}^2\left(\frac{3}{2}\right) = -\frac{1}{9\pi^2}$

$$Z(f) = \frac{1}{4}\delta(f) - \frac{1}{\pi^2}\left[\delta\left(f - \frac{2}{T}\right) + \delta\left(f + \frac{2}{T}\right)\right]$$

$$z(t) = \frac{1}{4} - \frac{2}{\pi^2}\cos\left(\frac{4\pi t}{T}\right)$$

COMPITO 19/2/2018 - FILA A

Es. 1 - Un sistema LTI discreto è caratterizzato dalla seguente equazione alle differenze:

$$y(n) + \frac{3}{4} y(n-1) - \frac{7}{64} y(n-2) = x(n-1) + x(n-2).$$

1) Si scriva la funzione di trasferimento del sistema e si faccia il grafico della forma canonica.

2) Si scrivano tutte le possibili espressioni della risposta impulsiva del sistema e le relative zone di convergenza.

Si consideri ora il sistema causale.

3) Si calcoli e si grafichi il modulo $|\bar{H}(f)|$ della risposta in freq. del sistema e, in base ad esso, si dica se il sistema è un passa-basso, passa-alto o passa-banda.

4) Si scriva l'espressione $y(n)$ della risposta del sistema alla sequenza $x(n) = u(n)$.

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1) $y(n) + \frac{3}{4} y(n-1) - \frac{7}{64} y(n-2) = x(n-1) + x(n-2)$

da cui

$$Y(z) + \frac{3}{4} z^{-1} Y(z) - \frac{7}{64} z^{-2} Y(z) = z^{-1} X(z) + z^{-2} X(z$$

$$\Rightarrow H(z) = \frac{z^{-1} + z^{-2}}{1 + \frac{3}{4} z^{-1} - \frac{7}{64} z^{-2}} = \frac{z+1}{z^2 + \frac{3}{4} z - \frac{7}{64}}$$

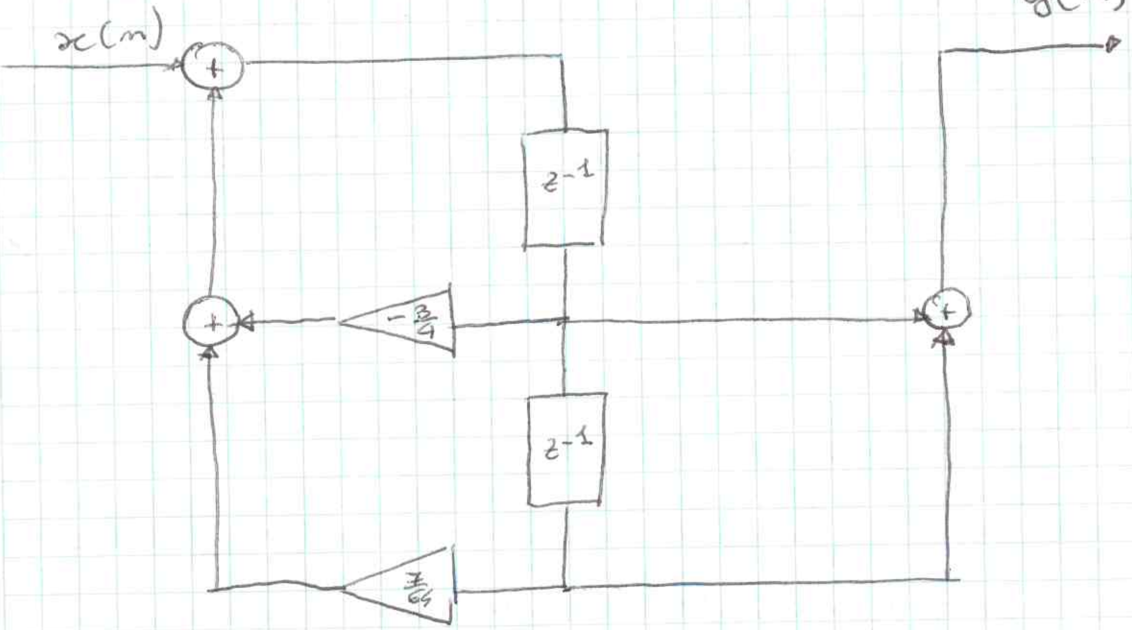
$$z_1 = \frac{1}{8} \qquad z_2 = -\frac{7}{8}$$

$$\Rightarrow H(z) = \frac{z+1}{\left(z - \frac{1}{8}\right)\left(z + \frac{7}{8}\right)}$$

Forma canonica

$$y(n) = -\frac{3}{4}\, y(n-1) + \frac{7}{64}\, y(n-2) + x(n-1) + x(n-2)$$

Grafico




2) Sviluppiamo la funz. di trasferimento in fratti parziali

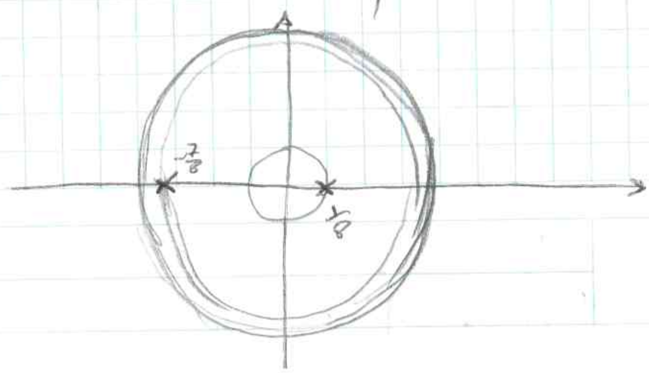
$$H(z) = \frac{A_1}{z-\frac{1}{8}} + \frac{A_2}{z+\frac{7}{8}}$$

$$A_1 = \left.\frac{z+1}{z+\frac{7}{8}}\right|_{z=\frac{1}{8}} = \frac{9}{8} \; ; \quad A_2 = \left.\frac{z+1}{z-\frac{1}{8}}\right|_{z=-\frac{7}{8}} = -\frac{1}{8}$$

$$H(z) = \frac{9}{8}\,\frac{1}{z-\frac{1}{8}} - \frac{1}{8}\,\frac{1}{z+\frac{7}{8}}$$

I poli del sistema sono 2, come nel grafico.

$p_1 = \frac{1}{8}$, $p_2 = -\frac{7}{8}$



a) $|z| < \frac{1}{8}$

$$h(m) = -\frac{9}{8}\left(\frac{1}{8}\right)^{m-1} u(-m) + \frac{1}{8}\left(-\frac{7}{8}\right)^{m-1} u(-m)$$

b) $\frac{1}{8} < |z| < \frac{7}{8}$

$$h(m) = \frac{9}{8}\left(\frac{1}{8}\right)^{m-1} u(m-1) + \frac{1}{8}\left(-\frac{7}{8}\right)^{m-1} u(-m)$$

c) $|z| > \frac{7}{8}$

$$h(m) = \frac{9}{8}\left(\frac{1}{8}\right)^{m-1} u(m-1) - \frac{1}{8}\left(-\frac{7}{8}\right)^{m-1} u(m-1)$$

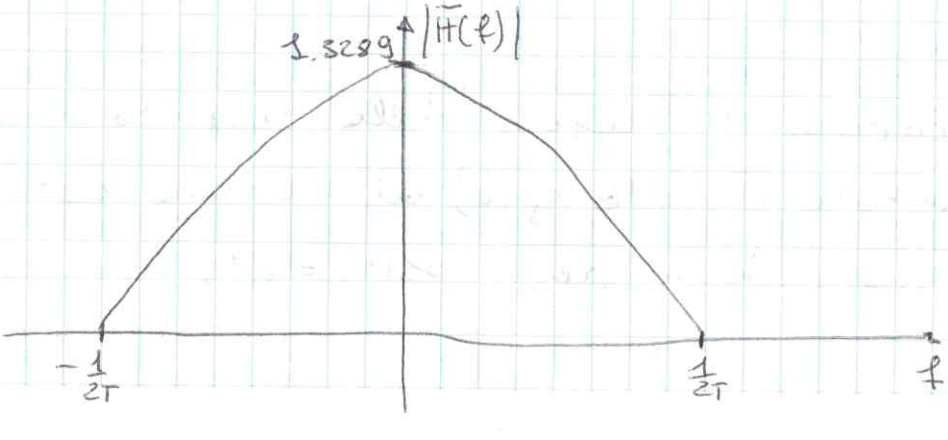
3) $$\overline{H}(f) = \frac{e^{j2\pi fT} + 1}{\left(e^{j2\pi fT} - \frac{1}{8}\right)\left(e^{j2\pi fT} + \frac{7}{8}\right)}$$

$$|\overline{H}(f)| = \frac{\sqrt{(\cos 2\pi fT + 1)^2 + \sin^2 2\pi fT}}{\sqrt{\left[\left(\cos 2\pi fT - \frac{1}{8}\right)^2 + \sin^2 2\pi fT\right]\left[\left(\cos 2\pi fT + \frac{7}{8}\right)^2 + \sin^2 2\pi fT\right]}}$$

$$= \frac{\sqrt{2 + 2\cos 2\pi fT}}{\sqrt{\left(\frac{7}{8} - \frac{2}{8}\cos 2\pi fT\right)\left(\frac{15}{8} + \frac{7}{4}\cos 2\pi fT\right)}}$$

$$|\overline{H}(0)| = \frac{2}{\sqrt{\frac{5}{8}\cdot\frac{29}{8}}} \simeq 1.3287 \qquad \left|H\left(\pm\frac{1}{2T}\right)\right| = 0$$

Filtro passa-basso non molto selettivo



4) $Y(z) = H(z)\dfrac{z}{z-1} = \dfrac{z(z+1)}{(z-1)(z-\frac{1}{8})(z+\frac{7}{8})}$

$= \dfrac{A_1}{z-1} + \dfrac{A_2}{z-\frac{1}{8}} + \dfrac{A_3}{z+\frac{7}{8}}$

$A_1 = Y(z)(z-1)\Big|_{z=1} = \dfrac{128}{105}$

$A_2 = Y(z)\left(z-\frac{1}{8}\right)_{z=\frac{1}{8}} = -\dfrac{9}{56}$

$A_3 = Y(z)\left(z+\frac{7}{8}\right)\Big|_{z=-\frac{7}{8}} = -\dfrac{7}{120}$

$Y(z) = \dfrac{128}{105}\dfrac{1}{z-1} - \dfrac{9}{56}\dfrac{1}{z-\frac{1}{8}} - \dfrac{7}{120}\dfrac{1}{z+\frac{7}{8}}$

$y(n) = \dfrac{128}{105}u(n-1) - \dfrac{9}{56}\left(\dfrac{1}{8}\right)^{n-1}u(n-1) - \dfrac{7}{120}\left(-\dfrac{7}{8}\right)^{n-1}u(n-1)$

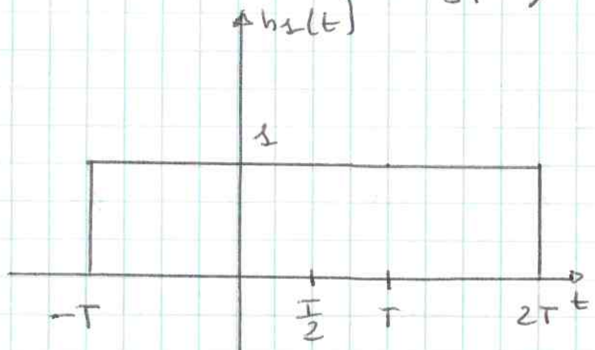
Es. 2 – Siano dati 2 sistemi LTI in serie. Il primo è caratterizzato da una risposta impulsiva $h_1(t) = \text{rect}\left(\dfrac{t-0.5T}{3T}\right)$, il secondo da $h_2(t) = \text{rect}\left(\dfrac{t-0.5T}{T}\right) - T\delta(t-4T)$.

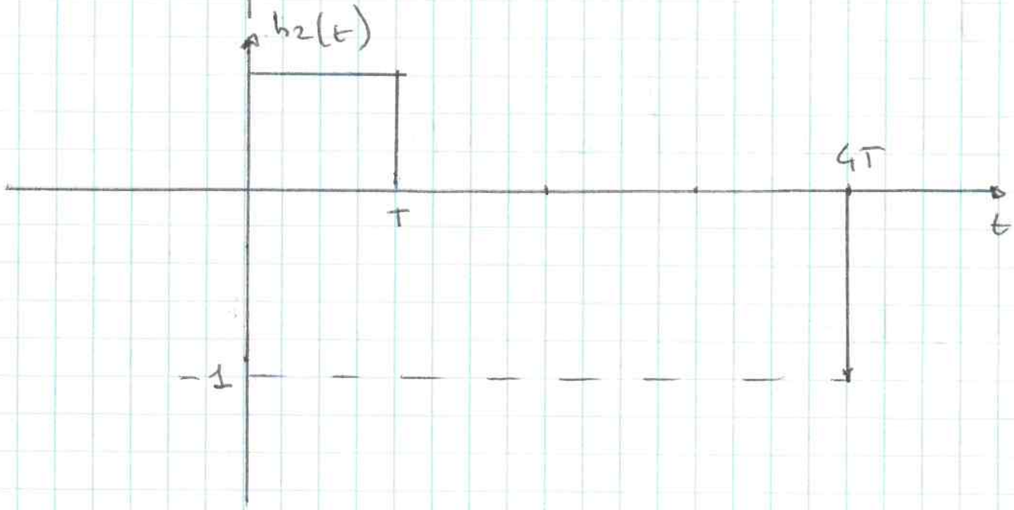
1) Si calcolino la risposta in frequenza $H_{eq}(f)$ e la risposta impulsiva $h_{eq}(t)$ equivalenti dell'intero sistema.

2) Si faccia il grafico della risposta impulsiva $h_{eq}(t)$.

3) Si calcoli l'uscita $y(t)$ del sistema corrispondente al segnale di ingresso $x(t) = u(t)$.

1) $h_1(t) = rect\left(\frac{t - T/2}{3T}\right)$ $h_2(t) = rect\left(\frac{t-T/2}{T}\right) - T\delta(t-4T)$




$h_{eq}(t) = h_1(t) \otimes h_2(t)$

$H_{eq}(f) = H_1(f)\ H_2(f)$

$H_1(f) = 3T\, sinc(3fT)\, e^{-j\pi fT}$

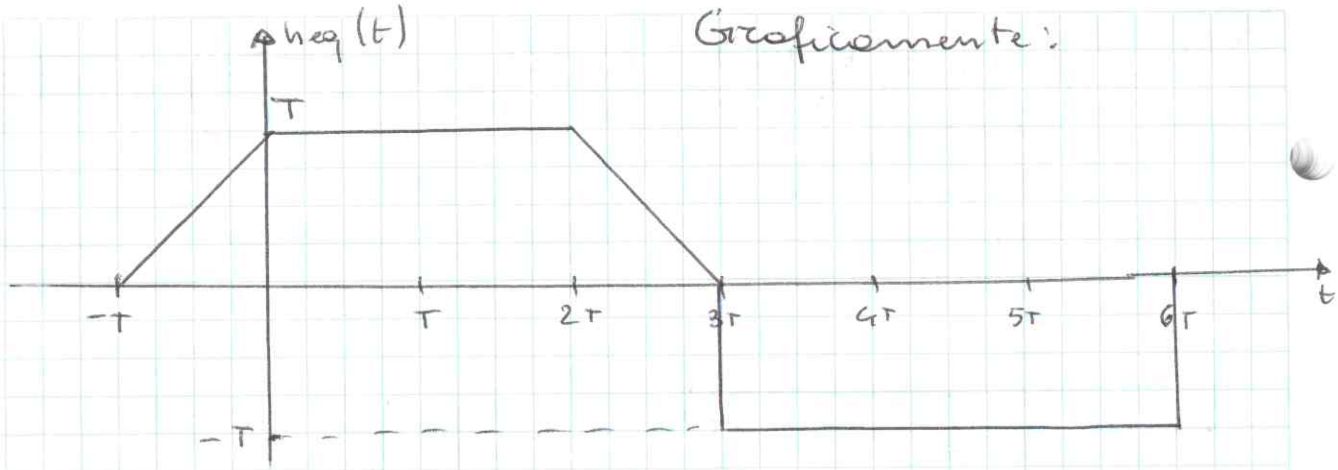
$H_2(f) = T\, sinc(fT)\, e^{-j\pi fT} - T e^{-j8\pi fT}$

$$H_{eq}(f) = 3T^2 sinc(3fT)\, sinc(fT)\, e^{-j2\pi fT} - 3T^2 sinc(3fT)\, e^{-j8\pi fT}$$

$$h_{eq}(t) = rect\left(\frac{t-T/2}{3T}\right) \otimes \left[rect\left(\frac{t-T/2}{T}\right) - T\delta(t-4T)\right]$$

$$= rect\left(\frac{t-T/2}{3T}\right) \otimes rect\left(\frac{t-T/2}{T}\right) - T\, rect\left(\frac{t-9/2T}{3T}\right)$$

Graficamente:




3) $x(t) = u(t)$

$$y(t) = u(t) \otimes h_{eq}(t) = \int_{-\infty}^{+\infty} h_{eq}(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-T}^{t} h_{eq}(\tau)\, d\tau$$

Dal grafico:

$$h_{eq}(t) = \begin{cases} T\left(1+\frac{t}{T}\right) & -T \le t \le 0 \\ T & 0 \le t \le 2T \\ T\left(3-\frac{t}{T}\right) & 2T \le t < 3T \\ -T & 3T \le t < 6T \\ 0 & \text{altrove} \end{cases}$$

$-T < t \le 0 \qquad y(t) = \int_{-T}^{t} T\left(1+\frac{\tau}{T}\right) d\tau = \frac{t^2}{2} + Tt + \frac{T^2}{2}$

$0 \le t \le 2T \qquad y(t) = \frac{T^2}{2} + \int_{0}^{t} T\, d\tau = \frac{T^2}{2} + Tt$

$2T \le t < 3T$

$$y(t) = \frac{T^2}{2} + 2T^2 + \int_{2T}^{t} T\left(3 - \frac{z}{T}\right) dz$$

$$= \frac{5}{2}T^2 + 3Tt - \frac{3}{2}t^2$$

$3T \le t < 6T$

$$y(t) = \frac{T^2}{2} + 2T^2 + \frac{T^2}{2} - \int_{3T}^{t} T \, dz$$

$$= 3T^2 - Tt$$

$t \ge 6T$

$$y(t) = \frac{T^2}{2} + 2T^2 + \frac{T^2}{2} - 3T^2 = 0$$

Riassumendo

$$y(t) = \begin{cases} 0 & t \le -T \\ \frac{t^2}{2} + Tt + \frac{T^2}{2} & -T \le t \le 0 \\ \frac{T^2}{2} + Tt & 0 \le t \le 2T \\ \frac{5}{2}T^2 + 3Tt - \frac{3}{2}t^2 & 2T \le t < 3T \\ 3T^2 - Tt & 3T \le t < 6T \\ 0 & t \ge 6T \end{cases}$$

FILA B

Stesso testo fila A, ma

$$y(n) + y(n-1) + \frac{2}{9}\, y(n-2) = x(n-1) - x(n-2)$$

$$Y(z) + z^{-1} Y(z) + \frac{2}{9}\, z^{-2} Y(z) = z^{-1} X(z) - z^{-2} X(z)$$

$$H(z) = \frac{z-1}{z^2 + z + \frac{2}{9}}$$

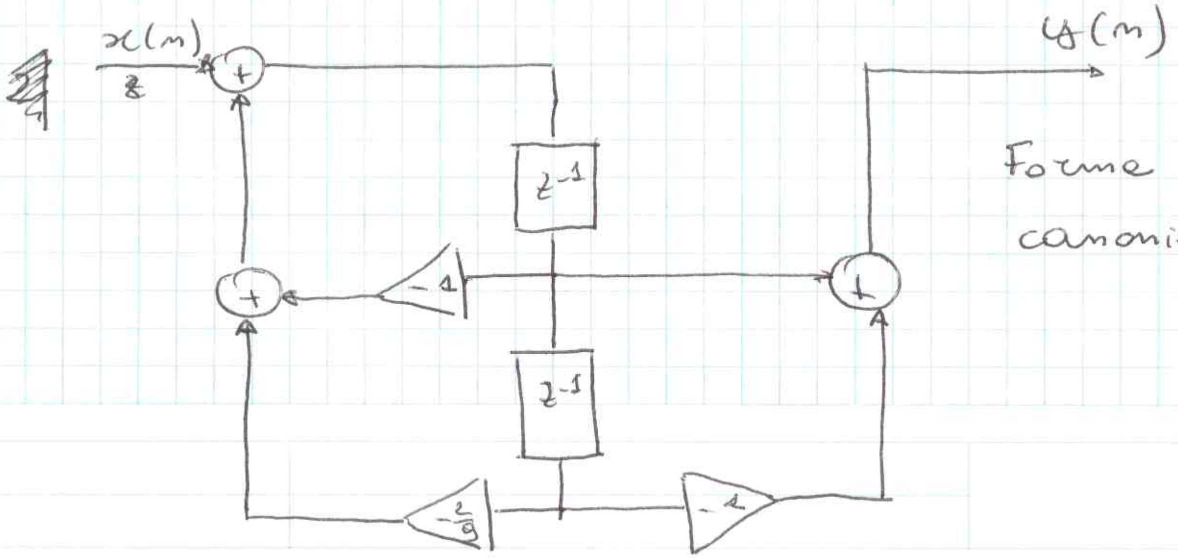
$$z_{1/2} = \frac{-1 \pm \sqrt{1 - 8/9}}{2} \quad \begin{cases} -\frac{2}{3} \\ -\frac{1}{3} \end{cases}$$

$$H(z) = \frac{z-1}{\left(z+\frac{2}{3}\right)\left(z+\frac{1}{3}\right)} = \frac{A_1}{z+\frac{2}{3}} + \frac{A_2}{z+\frac{1}{3}}$$

$$A_1 = \left.\frac{z-1}{z+\frac{1}{3}}\right|_{z=-\frac{2}{3}} = 5$$

$$A_2 = \left.\frac{z-1}{z+\frac{2}{3}}\right|_{z=-\frac{1}{3}} = -4$$

$$H(z) = \frac{5}{z+\frac{2}{3}} - \frac{4}{z+\frac{1}{3}}$$




Forma canonica

2) $|z| < \frac{1}{3}$ $\quad h(n) = -5\left(-\frac{2}{3}\right)^{n-1} u(-n) + 4\left(-\frac{1}{3}\right)^{n-1} u(-n)$

$\frac{1}{3} < |z| < \frac{2}{3}$ $\quad h(n) = -5\left(-\frac{2}{3}\right)^{n-1} u(-n) - 4\left(-\frac{1}{3}\right)^{n-1} u(n-1)$

$|z| > \frac{2}{3}$ $\quad h(n) = 5\left(-\frac{2}{3}\right)^{n-1} u(-n-1) - 4\left(-\frac{1}{3}\right)^{n-1} u(n-1)$

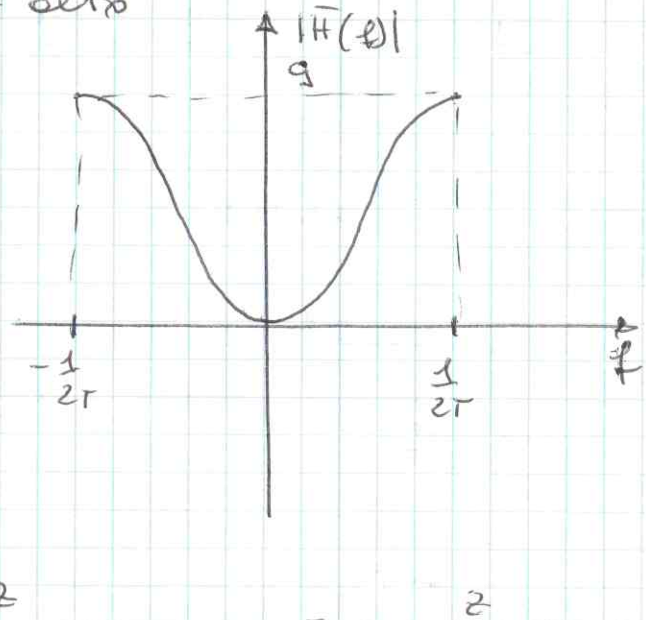
3) $$\overline{H}(f) = \frac{\left(e^{j2\pi fT} - 1\right)}{\left(e^{j2\pi fT} + \frac{2}{3}\right)\left(e^{j2\pi fT} + \frac{1}{3}\right)}$$

$$\left|\overline{H}(f)\right| = \frac{\sqrt{(\cos 2\pi fT - 1)^2 + \sin^2 2\pi fT}}{\sqrt{\left[\left(\cos 2\pi fT + \frac{2}{3}\right)^2 + \sin^2 2\pi fT\right]\left[\left(\cos 2\pi fT + \frac{1}{3}\right)^2 + \sin^2 2\pi fT\right]}}$$

$$= \frac{\sqrt{2 - 2\cos 2\pi fT}}{\sqrt{\left(\frac{13}{9} + \frac{4}{3}\cos 2\pi fT\right)\left(\frac{10}{9} + \frac{2}{3}\cos 2\pi fT\right)}}$$

$\left|\overline{H}(0)\right| = 0 \qquad \left|\overline{H}\left(\pm\frac{1}{2T}\right)\right| = 9$

Filtro passa-alto




4) $$Y(z) = \frac{\cancel{(z-1)}\, z}{\left(z+\frac{2}{3}\right)\left(z+\frac{1}{3}\right)\cancel{(z-1)}} = \frac{z}{\left(z+\frac{2}{3}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{A_1}{z+\frac{2}{3}} + \frac{A_2}{z+\frac{1}{3}}$$

$$A_1 = \left.\frac{z}{z+\frac{1}{3}}\right|_{z=-\frac{2}{3}} = 2 \quad ; \quad A_2 = \left.\frac{z}{z+\frac{2}{3}}\right|_{z=-\frac{1}{3}} = -1$$

$$Y(z) = \frac{2}{z+\frac{2}{3}} - \frac{1}{z+\frac{1}{3}}$$ da cui:

$$y(n) = 2\left(-\frac{2}{3}\right)^{n-1} u(n-1) - \left(-\frac{1}{3}\right)^{n-1} u(n-1)$$

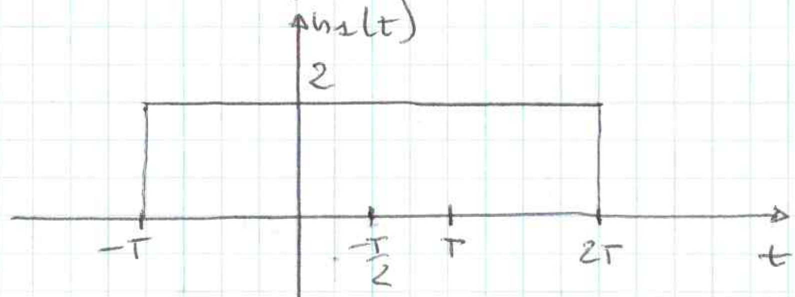
---

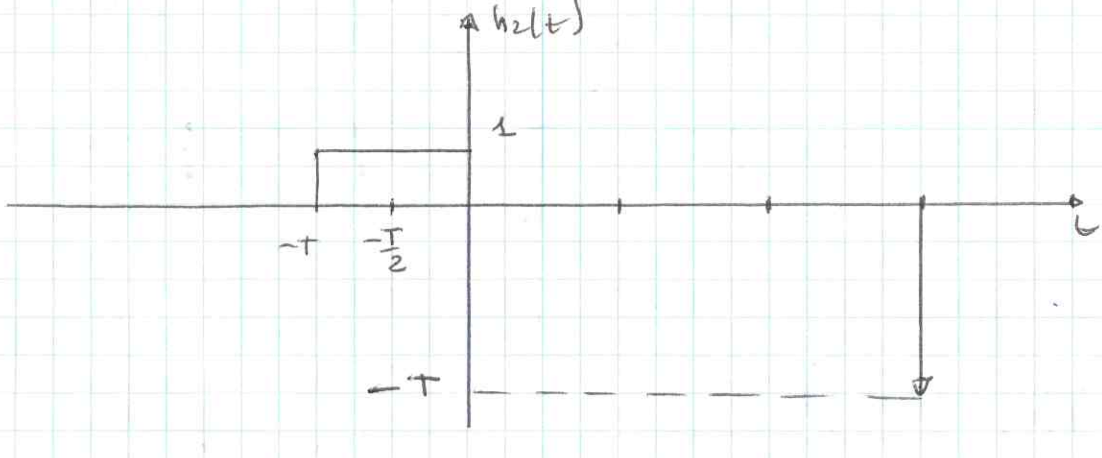
Es. 2 - Fila B.

Stesso testo ma $h_1(t) = 2\,\mathrm{rect}\left(\frac{t-0.5T}{3T}\right)$

e $h_2(t) = \mathrm{rect}\left(\frac{t+0.5T}{T}\right) - T\delta(t-3T)$

1)

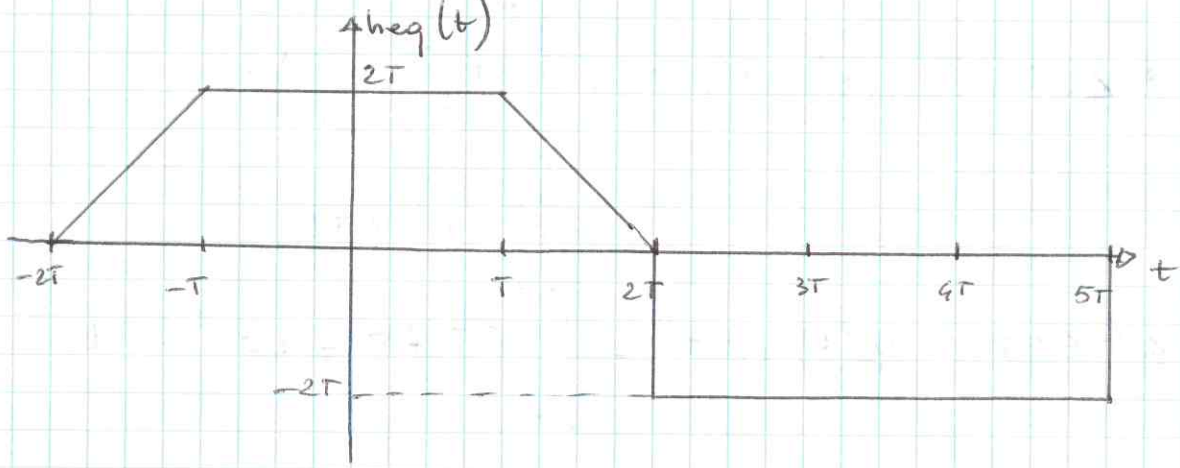


$$h_{eq}(t) = h_1(t) \otimes h_2(t) =$$

$$= 2\,\mathrm{rect}\left(\frac{t - T/2}{3T}\right) \otimes \mathrm{rect}\left(\frac{t + T/2}{T}\right) - 2T\,\mathrm{rect}\left(\frac{t - \frac{7}{2}T}{3T}\right)$$

2) Graficamente:




$$H_{eq}(f) = H_1(f)\,H_2(f)$$

$$= 6T\,\mathrm{sinc}(3fT)\,e^{-j\pi fT}\left[T\,\mathrm{sinc}(fT)\,e^{j\pi fT} - Te^{-j6\pi fT}\right]$$

$$= 6T^2\,\mathrm{sinc}(3fT)\,\mathrm{sinc}(fT) - 6T^2\,\mathrm{sinc}(3fT)\,e^{-j7\pi fT}$$

3) $y(t) = h_{eq}(t) \otimes u(t) = \displaystyle\int_{-2T}^{t} h_{eq}(\tau)\,d\tau$

$$h_{eq}(t) = \begin{cases} 2T\left(2 + \frac{t}{T}\right) & -2T \le t \le -T \\ 2T & -T \le t \le T \\ 2T\left(2 - \frac{t}{T}\right) & T \le t < 2T \\ -2T & 2T \le t < 5T \\ 0 & \text{altrove} \end{cases}$$

Integriamo per ottenere $y(t)$.

$$\begin{cases} -2T \le t \le -T & y(t) = \int_{-2T}^{t} 2T\left(\frac{\tau}{T}+2\right)d\tau = t^2+4Tt+4T^2 \\ -T \le t \le T & y(t) = T^2 + \int_{-T}^{t} 2T\,d\tau = 3T^2+2Tt \\ T \le t \le 2T & y(t) = T^2+4T^2+\int_{T}^{t} 2T\left(2-\frac{\tau}{T}\right)d\tau \\ & \quad = 2T^2-t^2+4Tt \\ 2T \le t \le 5T & y(t) = 6T^2 - \int_{2T}^{t} 2T\,d\tau = 10T^2-2Tt \end{cases}$$

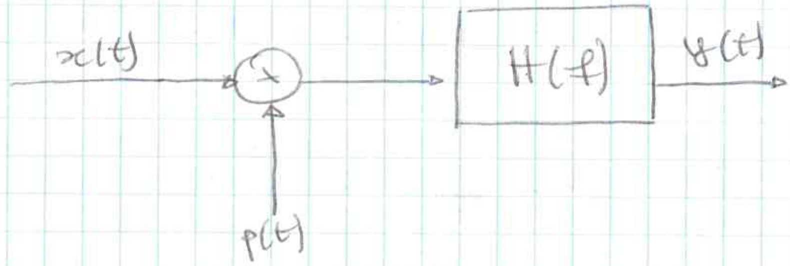
altrove $\quad 0$

Riassumendo

$$y(t) = \begin{cases} 0 & t \le -2T \\ t^2+4Tt+4T^2 & -2T \le t \le -T \\ 3T^2+2Tt & -T \le t \le T \\ 2T^2-t^2+4Tt & T \le t \le 2T \\ 10T^2-2Tt & 2T \le t \le 5T \\ 0 & t \ge 5T \end{cases}$$

COMPITO 7/4/18 - Fila B

Es. 1 - Sia dato il sistema in fig. 1

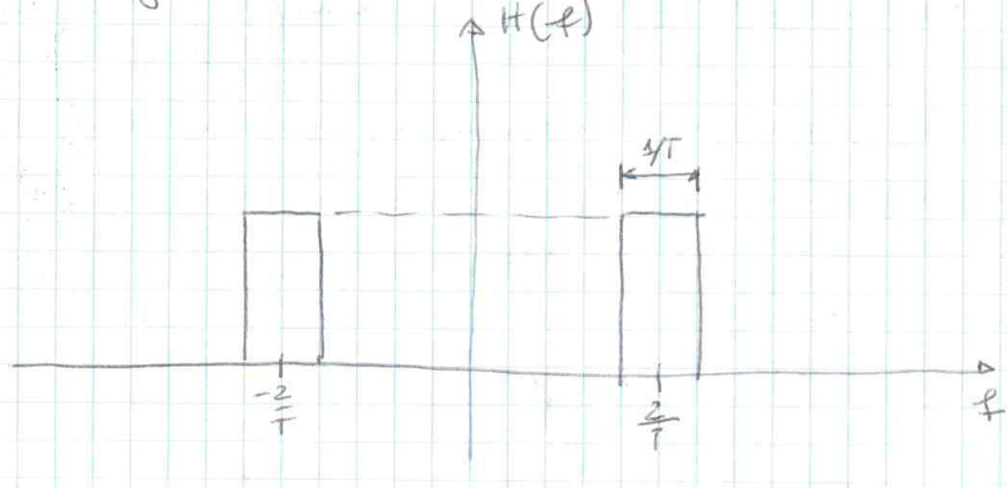



Il segnale di ingresso è $x(t)=\frac{1}{T}\,\mathrm{sinc}^2\left(\frac{t}{2T}\right)$. Il segnale $p(t)$ è così definito

$$p(t)=\sum_{m=-\infty}^{+\infty}\left[\mathrm{rect}\left(\frac{t-T/8-mT}{T/4}\right)+\mathrm{rect}\left(\frac{t-5T/8-mT}{T/4}\right)\right]$$

1) Dopo aver disegnato $p(t)$, si calcoli lo spettro del segnale $z(t)=x(t)\,p(t)$ e se ne faccia il grafico del modulo.

2) Il filtro $H(f)$ è un passa-banda ideale come in fig. Si calcoli lo spettro e l'andamento temporale del segnale all'uscita $y(t)$.

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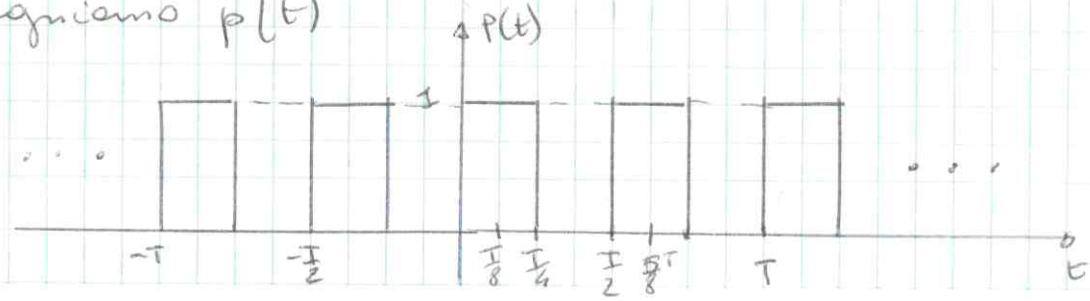
1) Disegniamo $p(t)$



Dalle figure si vede che il vero periodo di $p(t)$ è $T/2$, quindi

$$p(t) = \sum_{m=-\infty}^{+\infty} p_0\left(t - m\frac{T}{2}\right) \quad \text{con} \quad p_0(t) = rect\left(\frac{t - T/8}{T/4}\right)$$

$$P_0(f) = \mathcal{F}[p_0(t)] = \frac{T}{4}\, sinc\left(\frac{fT}{4}\right) e^{-j\pi f \frac{T}{4}}$$

$$P(f) = \frac{2}{T} \sum_{k=-\infty}^{+\infty} P_0\left(\frac{2k}{T}\right) \delta\left(f - \frac{2k}{T}\right)$$

$$= \frac{2}{T} \sum_{k=-\infty}^{+\infty} \frac{T}{4}\, sinc\left(\frac{2k}{T} \cdot \frac{T}{4}\right) e^{-j\pi \frac{k2}{T} \cdot \frac{T}{4}} \delta\left(f - \frac{2k}{T}\right)$$

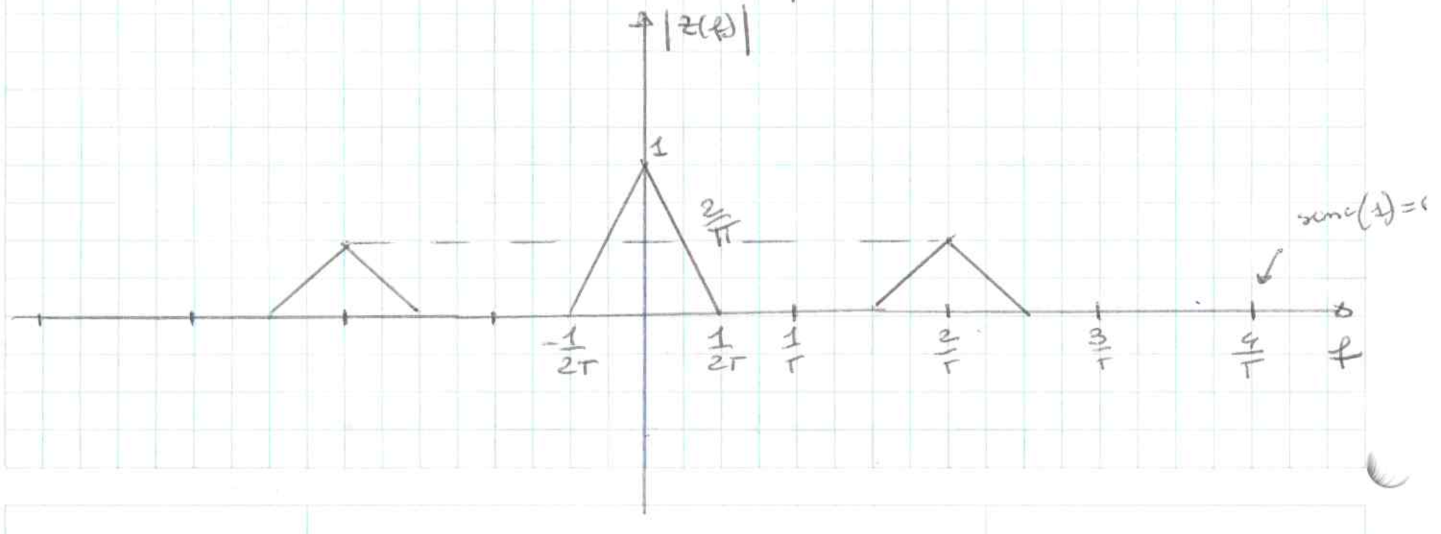
$$= \frac{1}{2} \sum_{k=-\infty}^{+\infty} sinc\left(\frac{k}{2}\right) e^{-j\frac{k\pi}{2}} \delta\left(f - \frac{2k}{T}\right)$$

Calcoliamo ora la trasformata del segnale di ingresso $x(t)$

$$X(f) \doteq \mathcal{F}[x(t)] = 2\, tr\left(\frac{f}{1/T}\right)$$

$$Z(f) = X(f) \otimes P(f) = \sum_{k=-\infty}^{+\infty} sinc\left(\frac{k}{2}\right) e^{-j\frac{\pi k}{2}}\, tr\left(\frac{f - \frac{2k}{T}}{1/T}\right)$$

Le repliche non si sovrappongono poiché sono larghe $1/T$ e centrate in multipli di $\frac{2}{T}$.



2) Il filtro ideale passa-banda seleziona le repliche a $\pm\frac{2}{T}$, quindi

$$Y(f) = Z(f)H(f) = -i\frac{2}{\pi}\, tr\left(\frac{f-\frac{2}{T}}{1/T}\right) + i\frac{2}{\pi}\, tr\left(\frac{f+\frac{2}{T}}{1/T}\right)$$

Antitrasformiamo

$$y(t) = \mathcal{F}^{-1}\left[Y(f)\right] = -i\frac{2}{\pi}\cdot\frac{1}{2T}\, sinc^2\left(\frac{t}{2T}\right) e^{2i\pi\frac{2}{T}t}$$

$$+ i\frac{2}{\pi}\,\frac{1}{2T}\, sinc^2\left(\frac{t}{2T}\right) e^{-2i\pi\frac{2}{T}t}$$

$$= -i\left[e^{2i\pi\frac{2}{T}t} - e^{-2i\pi\frac{2}{T}t}\right]\frac{1}{\pi T}\, sinc^2\left(\frac{t}{2T}\right)$$

$$\boxed{= \frac{2}{\pi T}\, \sin\left(\frac{4\pi t}{T}\right) sinc^2\left(\frac{t}{2T}\right)}$$

---

Es. 2 — Sia dato il seguente filtro analogico causale $H_a(s) = \frac{1}{s+k/T_0}$

1) Si calcolino la risposta in freq. e la banda a $-3$dB del filtro con $T_0 > 0$ e $k > 0$.

2) A partire dal filtro analogico si vuole progettare un filtro numerico passa-basso. A tal fine si utilizza l'invarianza della risp. impulsiva. Supponendo di campionare con $T = T_0/10$ si scriva la $H(z)$ del nuovo filtro e se ne individui la zona di co

3) Si calcoli il modulo della risposta in freq. del filtro num. e, dopo averne calcolato il valore in $f=0$ e $f=\pm 1/2T$, se ne faccia il grafico.

4) Si applichi ora al filtro analogico la trasf. bil. con $T=T_0/10$. Si scrivano la funz. di trasferimento del nuovo filtro num. e l'espressione del modulo della risposta in freq.

5) Quanto deve valere k affinché la banda a $-3$dB del filtro digitale sia pari a $1/T_0$?

1) $H_a(s) = \dfrac{1}{s+\frac{k}{T_0}}$

Calcoliamo la risposta in frequenza

$H_a(f) = \dfrac{1}{j2\pi f + \frac{k}{T_0}} \Rightarrow |H_a(f)| = \dfrac{1}{\sqrt{4\pi^2 f^2 + \frac{k^2}{T_0^2}}}$

Calcoliamo la banda a $-3\,dB$ ricordando che

$\dfrac{|H(B_{-3})|^2}{|H(0)|^2} = \dfrac{1}{2}$ $\quad |H(0)|^2 = \dfrac{T_0^2}{k^2}$

$\Rightarrow \boxed{B_{-3} = \dfrac{1}{2\pi}\dfrac{k}{T_0}}$

2) Applichiamo l'invarianza della risposta impulsiva con $T = \dfrac{T_0}{10}$

Antitrasformando $H_a(s)$ otteniamo

$h_a(t) = e^{-\frac{k}{T_0}t}\, u(t)$

Campionando e moltiplicando per T

$h(nT) = T e^{-\frac{k}{T_0}nT} u(n)$ per $T = \dfrac{T_0}{10}$

$\Rightarrow h(n) = T e^{-n\frac{k}{10}} u(n)$

Osserviamo che in $\phi$ si ha una discontinuità di prima specie. Per far sì che $h(0) = \dfrac{h(0^+)+h(0^-)}{2}$ dobbiamo aggiungere un termine correttivo

$\Rightarrow h(n) = T e^{-n\frac{k}{10}} u(n) - \dfrac{T}{2}\delta(n)$

Di conseguenza

$H(z) = \dfrac{T}{1 - e^{-\frac{k}{10}} z^{-1}} - \dfrac{T}{2} =$

$$= \frac{T}{2} \frac{1+e^{-\frac{k}{T_0}} z^{-1}}{1 - e^{-\frac{k}{T_0}} z^{-1}} = \frac{T}{2} \frac{z+e^{-\frac{k}{T_0}}}{z - e^{-\frac{k}{T_0}}}$$

la zona di convergenza $|z| > e^{-\frac{k}{T_0}}$

3) Calcoliamo ora la risposta in frequenza del sistema. La circonferenza di raggio unitario fa parte della zona di convergenza, dunque

$$\overline{H}(f) = \frac{T}{2} \frac{e^{j2\pi f T} + e^{-\frac{k}{T_0}}}{e^{j2\pi f T} - e^{-\frac{k}{T_0}}}$$

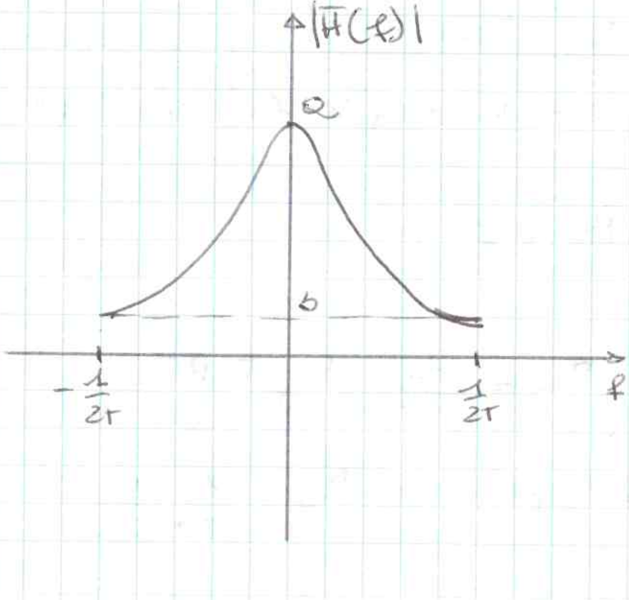
$$|\overline{H}(f)| = \frac{T}{2} \frac{\sqrt{\left(\cos 2\pi f T + e^{-\frac{k}{T_0}}\right)^2 + \sin^2 2\pi f T}}{\sqrt{\left(\cos 2\pi f T - e^{-\frac{k}{T_0}}\right)^2 + \sin^2 2\pi f T}}$$

$$\boxed{= \frac{T}{2} \frac{\sqrt{1+e^{-\frac{k}{5}} + 2e^{-\frac{k}{T_0}} \cos 2\pi f T}}{\sqrt{1+e^{-\frac{k}{5}} - 2e^{-\frac{k}{T_0}} \cos 2\pi f T}}}$$

$$|\overline{H}(0)| = \frac{T}{2} \frac{1+e^{-\frac{k}{T_0}}}{1-e^{-\frac{k}{T_0}}} = a$$

$$\left|\overline{H}\left(\pm \frac{1}{2T}\right)\right| = \frac{T}{2} \frac{1-e^{-\frac{k}{T_0}}}{1+e^{-\frac{k}{T_0}}} = b \quad \text{con } b < a$$

Il filtro è un passa-basso.



4) Applichiamo ora la trasformazione bilineare con $T = T_0/10$

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\Rightarrow H(z) = \frac{1}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \frac{k}{T_0}}$$

$$= \frac{10T(1+z^{-1})}{(20+k)+(k-20)z^{-1}}$$

Calcoliamo la risposta in frequenza

$$\bar{H}(f) = \frac{10T(1+e^{-j2\pi fT})}{(20+k)+(k-20)e^{-j2\pi fT}} =$$

$$= \frac{10T e^{-j\pi fT}\left(e^{j\pi fT}+e^{-j\pi fT}\right)}{(20+k)+(k-20)e^{-j2\pi fT}}$$

$$|\bar{H}(f)| = \frac{20T|\cos\pi fT|}{\sqrt{\left[(20+k)+(k-20)\cos 2\pi fT\right]^2+(k-20)^2\sin^2 2\pi fT}}$$

$$= \frac{20T|\cos\pi fT|}{\sqrt{800+2k^2+2(k^2-400)\cos 2\pi fT}}$$

5) $k = ? \Rightarrow B = B_d = \frac{1}{T_0}$

Si sa che $B_a = \frac{1}{\pi T}\,tg\,\pi BT$

Dai calcoli del pt 1 $B_a = \frac{1}{2\pi}\frac{k}{T_0}$

dunque

$$\frac{1}{2\pi}\frac{K}{T_0} = \frac{1}{\pi T} \operatorname{tg} \pi B T$$

Per $T = T_0/10$ otteniamo

$$\frac{1}{2\pi}\frac{K}{\cancel{T_0}} = \frac{10}{\pi \cancel{T_0}} \operatorname{tg} \pi \frac{1}{\cancel{T_0}} \cdot \frac{\cancel{T_0}}{10}$$

$$\Rightarrow \boxed{K = 20 \operatorname{tg} \frac{\pi}{10}}$$

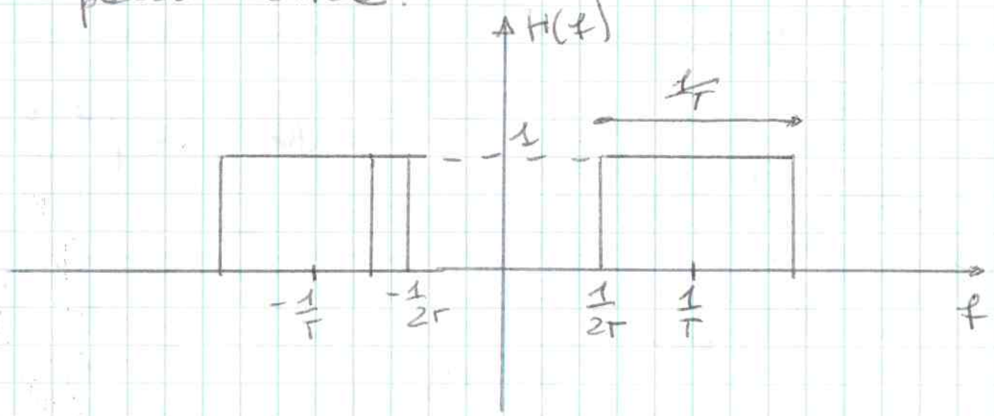
File A

Es 1. Stesso schema di fig. 1 ma ora il segnale

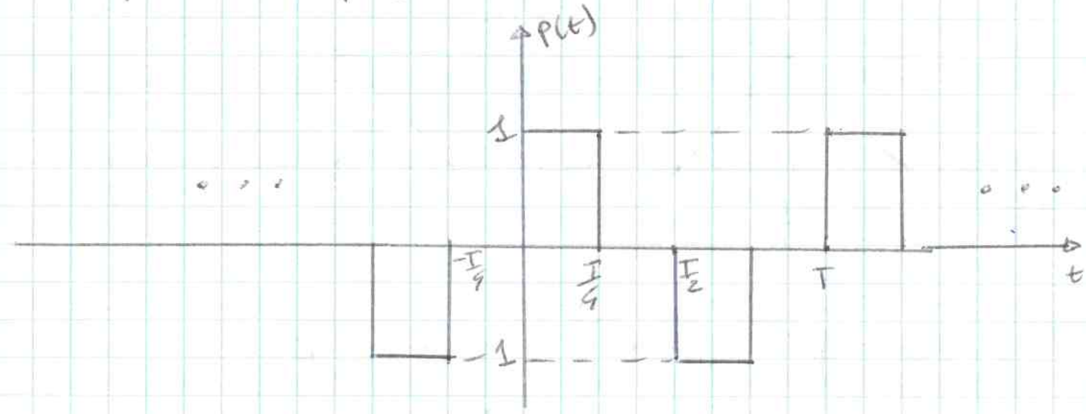
$$p(t) = \sum_{m=-\infty}^{+\infty} \left[ \operatorname{rect}\left(\frac{t - T/8 - mT}{T/4}\right) - \operatorname{rect}\left(\frac{t - 5T/8 - mT}{T/4}\right) \right]$$

1) Stesse domande file B

2) Stesse domande file B ma con il seguente filtro passa-banda:




1) Grafico di $p(t)$



$$p_0(t) = rect\left(\frac{t - T/8}{T/4}\right) - rect\left(\frac{t - 5T/8}{T/4}\right)$$

Il periodo è T.

$$P_0(f) = \frac{T}{4} sinc\left(\frac{fT}{4}\right) e^{-j\pi f \frac{T}{4}} - \frac{T}{4} sinc\left(\frac{fT}{4}\right) e^{-j\pi f \frac{5T}{4}}$$

$$= \frac{T}{4} sinc\left(\frac{fT}{4}\right) e^{-j\frac{3}{4}\pi f T} \left[ e^{j\frac{\pi f T}{2}} - e^{-j\frac{\pi f T}{2}} \right]$$

$$= \frac{T}{2} j \; sinc\left(\frac{fT}{4}\right) e^{-j\frac{3}{4}\pi f T} \sin\left(\frac{\pi f T}{2}\right)$$

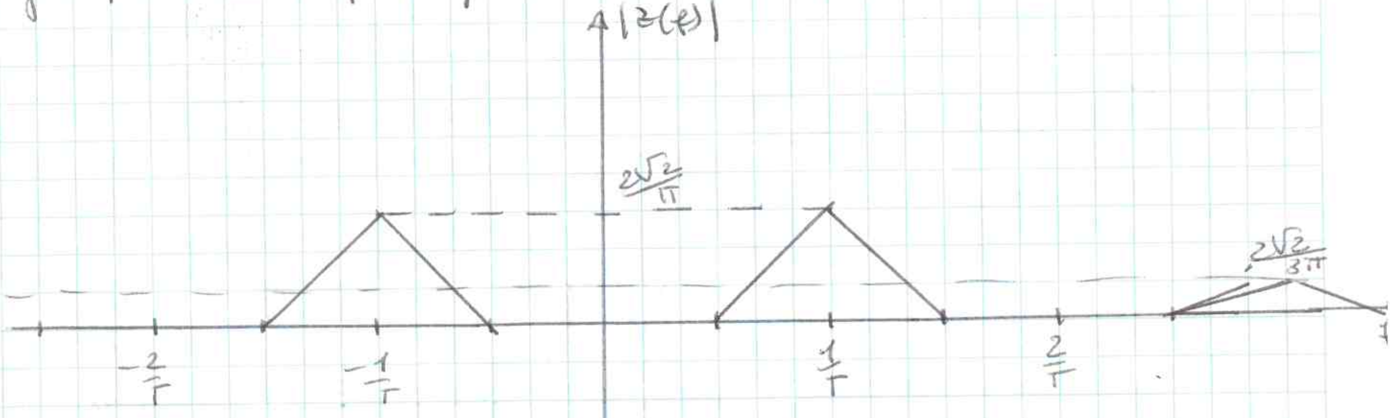
$$P(f) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} P_0\left(\frac{k}{T}\right) \delta\left(f - \frac{k}{T}\right)$$

$$= \frac{1}{2} j \sum_{k=-\infty}^{+\infty} sinc\left(\frac{k}{4}\right) e^{-j\frac{3\pi k}{4}} \sin\left(\frac{k\pi}{2}\right) \delta\left(f - \frac{k}{T}\right)$$

$$X(f) = 2 tr\left(\frac{f}{1/T}\right)$$

$$\Rightarrow Z(f) = j \sum_{k=-\infty}^{+\infty} sinc\left(\frac{k}{4}\right) e^{-j\frac{3\pi k}{4}} \sin\left(\frac{k\pi}{2}\right) tr\left(\frac{f - \frac{k}{T}}{1/T}\right)$$

Le repliche non si sovrappongono. Facciamo il grafico di $|Z(f)|$.




$k = 0 \qquad C_0 = j \, sinc\left(\frac{k}{4}\right) e^{-j\frac{3\pi k}{4}} \sin\left(\frac{k\pi}{2}\right) = 0$

$k = 1 \qquad C_1 = j \, sinc\left(\frac{1}{4}\right) \sin\left(\frac{\pi}{2}\right) e^{-j\frac{3\pi}{4}} = j \frac{2\sqrt{2}}{\pi} e^{-j\frac{3\pi}{4}}$

$k=2 \quad c_2=0$

$k=3 \quad c_3 = j\,\text{sinc}\left(\frac{3}{4}\right)\sin\left(\frac{3}{2}\pi\right) e^{-j\frac{9\pi}{4}} = -j\,\frac{2\sqrt{2}}{3\pi}\, e^{-j\frac{\pi}{4}}$

2) Applichiamo il filtro $H(f)$

$$Y(f) = Z(f)H(f) = j\,\text{sinc}\left(\frac{1}{4}\right) e^{-j\frac{3\pi}{4}} \sin\left(\frac{\pi}{2}\right) tr\left(\frac{f-\frac{1}{T}}{1/T}\right)$$
$$+ j\,\text{sinc}\left(-\frac{1}{4}\right) e^{j\frac{3\pi}{4}} \sin\left(-\frac{\pi}{2}\right) tr\left(\frac{f+1/T}{1/T}\right)$$

$$= \frac{2\sqrt{2}}{\pi} j \left[ e^{-j\frac{3\pi}{4}} tr\left(\frac{f-\frac{1}{T}}{1/T}\right) - e^{j\frac{3\pi}{4}} tr\left(\frac{f+\frac{1}{T}}{1/T}\right)\right]$$

$$= \frac{2\sqrt{2}}{\pi} j \left[ e^{-j\frac{3\pi}{4}} \frac{1}{2T} \text{sinc}^2\left(\frac{t}{2T}\right) e^{j2\pi t\frac{1}{T}} - e^{j\frac{3\pi}{4}} \frac{1}{2T} \text{sinc}^2\left(\frac{t}{2T}\right) e^{-j2\pi t\cdot\frac{1}{T}} \right]$$

$$= \frac{-2\sqrt{2}}{\pi T} \text{sinc}^2\left(\frac{t}{2T}\right) \sin\left(\frac{2\pi t}{T} - \frac{3\pi}{4}\right)$$

---

Es. 2. Sia dato il seguente filtro analogico causale

$H_a(s) = \frac{1}{s+k/T_0}$

1) Si calcolino la risp. in freq. e la banda a $-3$dB del filtro con $T_0>0$ e $k>0$.

A partire dal filtro analogico si vuole progettare un filtro num. passa-basso. A tal fine si utilizza l'inv. alla risposta imp.

2) Supponendo di campionare a $T=T_0/12$, si scriva la $H(z)$ del nuovo filtro e se ne individui la zona di convergenza.

3) Si calcoli $|\bar{H}(f)|$ e dopo averne calcolato i valori in $\emptyset$ e $\pm 1/2T$ se ne faccia il grafico

4) Si applichi ora la trasf. bilineare con $T=T_0/12$. Si scrivano

la funzione di trasf. del nuovo filtro num. e l'espressione di $|\bar{H}(f)|$.

5) Quanto deve valere $k$ affinché la banda a $-3dB$ del filtro digitale sia pari a $2/T_0$?

---

Stessa sol. della fila A

1) $B_{-3} = \frac{1}{2\pi}\frac{k}{T_0}$

2) $T = \frac{T_0}{12}$

$h(n) = T e^{-n\frac{k}{12}} u(n) - \frac{T}{2}\delta(n)$

$\Rightarrow H(z) = \frac{T}{2}\frac{z + e^{-\frac{k}{12}}}{z - e^{-\frac{k}{12}}} \qquad |z| > e^{-\frac{k}{12}}$

3) $|\bar{H}(f)| = \frac{T}{2}\frac{\sqrt{1+e^{-\frac{k}{6}} + 2e^{-\frac{k}{12}}\cos 2\pi fT}}{\sqrt{1+e^{-\frac{k}{6}} - 2e^{-\frac{k}{1}}\cos 2\pi fT}}$

Passa - basso

4) $H(z) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{k}{12}} = \frac{12T(1+z^{-1})}{(24+k)+(k-24)z^{-1}}$

$|\bar{H}(f)| = \frac{24T|\cos \pi fT|}{\sqrt{1152 + 2k^2 + 2(k-24)\cos 2\pi fT}}$

5) $k = ?$ $\qquad B = B_d = \frac{2}{T_0}$

$B_a = \frac{1}{\pi T}\operatorname{tg}\pi BT$

| | |
|---|---|
| Dal pt 1 | $B_a = \frac{1}{2\pi}\frac{k}{T_0}$ |

$$\frac{1}{2\pi}\frac{K}{T_0} = \frac{1}{\pi T}\, tg\, \pi B T$$

Per $T = T_0/12$ e $B = \frac{2}{T_0}$

$$K = \frac{2T_0}{T}\, tg\left(\pi \frac{2}{T_0} \cdot T\right) = 24\, tg\left(\frac{\pi}{6}\right)$$

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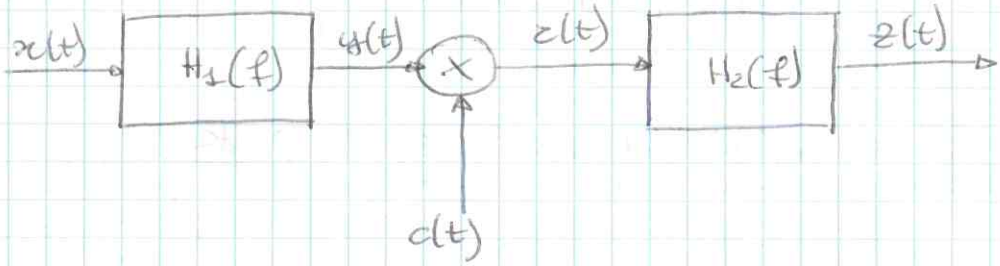
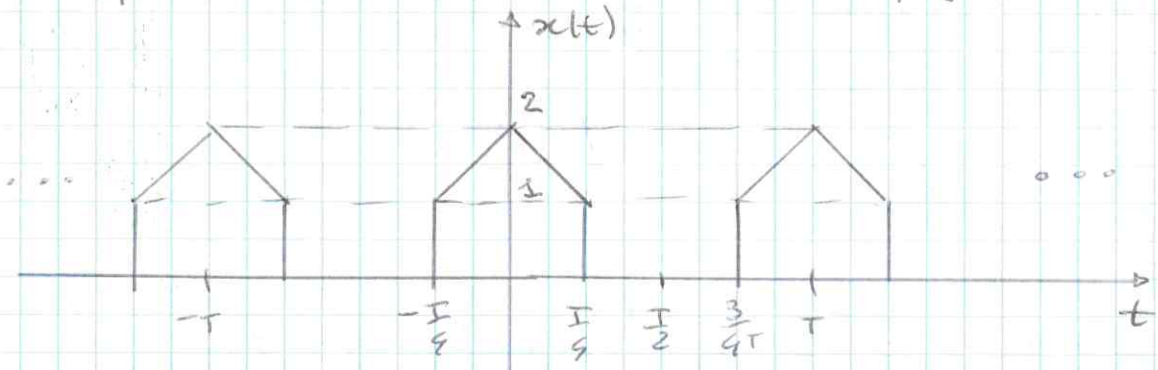
COMPITO 7/6/2018

Es. 1 – Sia dato il sistema in figura 1




dove $H_1(f) = tr\left(\frac{fT}{6}\right)$, $c(t) = \sum_{m=-\infty}^{+\infty} \delta\left(t - \frac{mT}{2}\right)$ e

$H_2(f) = \frac{T}{2} rect\left(\frac{fT}{5}\right)$. Il segnale $x(t)$ d'ingresso è periodico, di periodo $T$, come illustrato in fig. 2




1) Si calcoli lo spettro del segnale $x(t)$
2) Si calcoli ~~lo~~ spettro di $y(t)$ e se ne faccia il grafico.
3) Si calcoli lo spettro di $z(t)$ e si calcoli l'espressione temporale di $z(t)$.

1) Il segnale $x(t)$ può essere scritto come

$$x(t) = \sum_{k=-\infty}^{+\infty} \left[ rect\left(\frac{t-kT}{T/2}\right) + tr\left(\frac{t-kT}{T/2}\right) \right]$$

dove $rect\left(\frac{t}{T/2}\right) + tr\left(\frac{t}{T/2}\right) = x_0(t)$

$$X(f) = \sum_{n=-\infty}^{+\infty} X_k \, \delta\left(f - \frac{k}{T}\right) \quad \text{dove } X_k = \frac{1}{T} X_0\left(\frac{k}{T}\right)$$

$$X_0(f) = \mathcal{F}\left[x_0(t)\right] = \frac{T}{2} \operatorname{sinc}\left(\frac{fT}{2}\right) + \frac{T}{4} \operatorname{sinc}^2\left(\frac{fT}{4}\right)$$

$$X_k = \frac{1}{T} X_0\left(\frac{k}{T}\right) = \frac{1}{2} \operatorname{sinc}\left(\frac{k}{2}\right) + \frac{1}{4} \operatorname{sinc}^2\left(\frac{k}{4}\right)$$

$$\Rightarrow X(f) = \sum_{k=-\infty}^{+\infty} \left[\frac{1}{2} \operatorname{sinc}\left(\frac{k}{2}\right) + \frac{1}{4} \operatorname{sinc}^2\left(\frac{k}{4}\right)\right] \delta\left(f - \frac{k}{T}\right)$$

$$X_0 = \frac{3}{4} \; ; \quad X_1 = X_{-1} = \frac{1}{2} \operatorname{sinc}\left(\frac{1}{2}\right) + \frac{1}{4} \operatorname{sinc}^2\left(\frac{1}{4}\right) = \frac{1}{\pi} + \frac{2}{\pi^2}$$

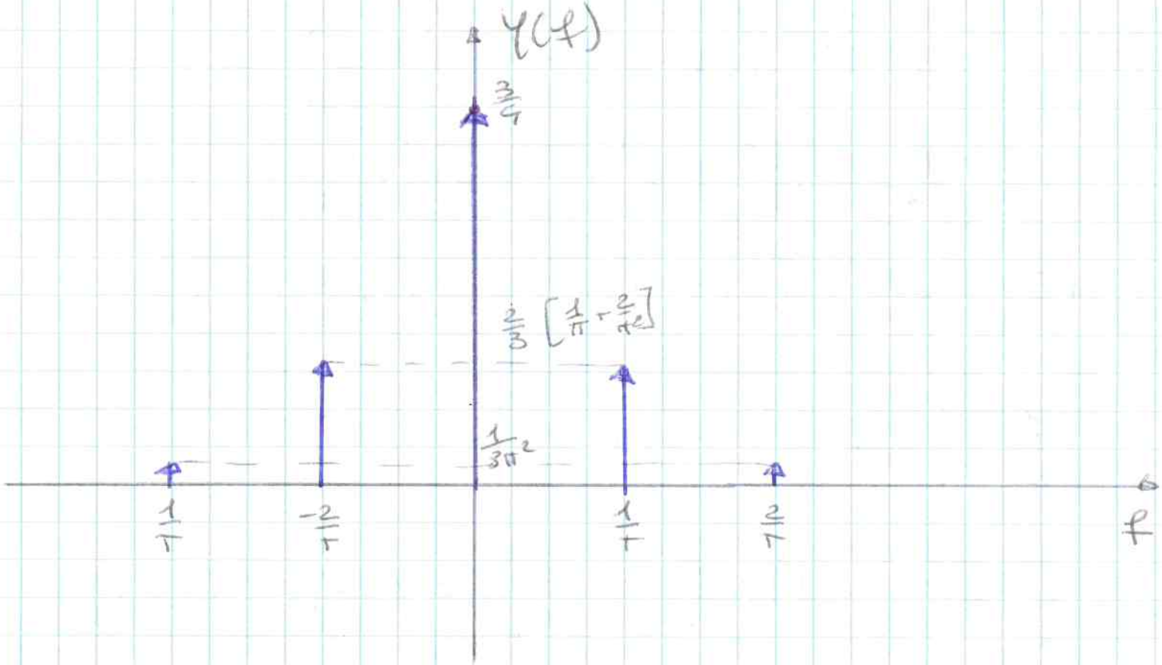
$$X_2 = X_{-2} = \frac{1}{2} \operatorname{sinc}(1) + \frac{1}{4} \operatorname{sinc}^2\left(\frac{1}{4}\right) = \frac{1}{\pi^2}$$

2) $$Y(f) = X(f) H_1(f) = \sum_{k=-\infty}^{+\infty} \left[\frac{1}{2} \operatorname{sinc}\left(\frac{k}{2}\right) + \frac{1}{4} \operatorname{sinc}^2\left(\frac{k}{4}\right)\right] \delta\left(f - \frac{k}{T}\right)$$

$$\cdot \, tr\left(\frac{fT}{4}\right)$$

$$= \frac{3}{4} \delta(f) + \frac{2}{3} \left[\frac{1}{\pi} + \frac{2}{\pi^2}\right] \left[\delta\left(f - \frac{1}{T}\right) + \delta\left(f + \frac{1}{T}\right)\right]$$

$$+ \frac{1}{3\pi^2} \left[\delta\left(f - \frac{2}{T}\right) + \delta\left(f + \frac{2}{T}\right)\right]$$




3) $r(t) = c(t) \cdot y(t)$

$$R(f) = C(f) \otimes Y(f)$$

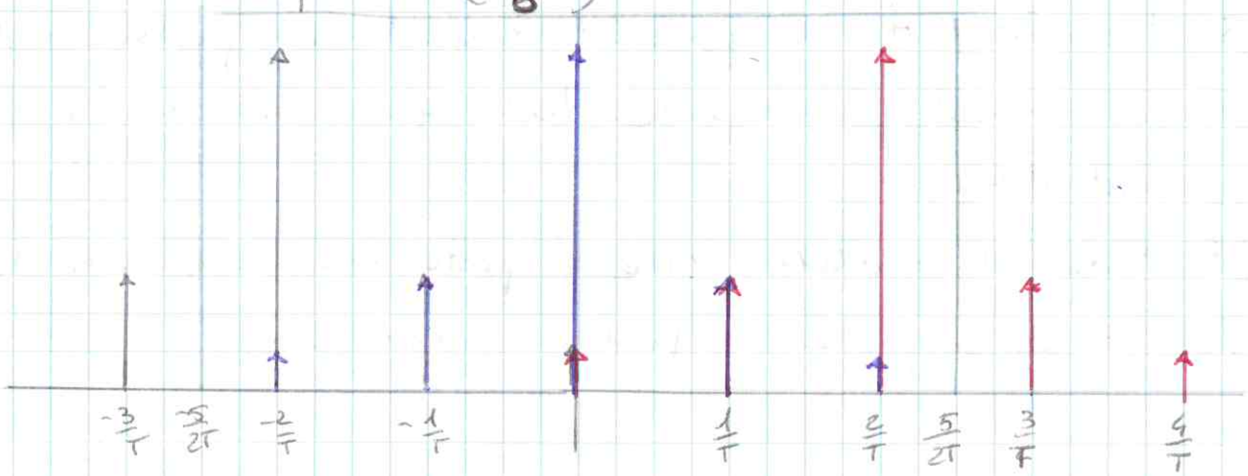
$$C(f) = \mathcal{F}[c(t)] = \frac{2}{T} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{2k}{T}\right)$$

$$\Rightarrow R(f) = \frac{2}{T} \sum_{k=-\infty}^{+\infty} Y\left(f - \frac{2k}{T}\right)$$

$$= \frac{2}{T} \sum_{k=-\infty}^{+\infty} \left[ \frac{3}{4} \delta\left(f - \frac{2k}{T}\right) + \frac{2}{3}\left(\frac{1}{\pi} + \frac{2}{\pi^2}\right)\left(\delta\left(f - \frac{1}{T} - \frac{2k}{T}\right) + \delta\left(f + \frac{1}{T} - \frac{2k}{T}\right)\right) \right.$$

$$\left. + \frac{1}{3\pi^2}\left(\delta\left(f - \frac{2}{T} - \frac{2k}{T}\right) + \delta\left(f + \frac{2}{T} - \frac{2k}{T}\right)\right) \right]$$

$$Z(f) = R(f) \cdot \frac{2}{T} \, \mathrm{rect}\left(\frac{fT}{5}\right)$$



$$z(f) = \frac{3}{4}\delta(f) + \frac{3}{4}\delta\left(f-\frac{2}{T}\right) + \frac{3}{4}\delta\left(f+\frac{2}{T}\right)$$

$$+\frac{2}{3}\left(\frac{1}{\pi}+\frac{2}{\pi^2}\right)\left[\delta\left(f-\frac{1}{T}\right)+\delta\left(f+\frac{1}{T}\right)\right]+\frac{1}{3\pi^2}\left[\delta\left(f-\frac{2}{T}\right)+\delta\left(f+\frac{2}{T}\right)\right]$$

$$+\frac{2}{3}\left(\frac{1}{\pi}+\frac{2}{\pi^2}\right)\left[\delta\left(f-\frac{1}{T}\right)+\delta\left(f+\frac{1}{T}\right)\right]+\frac{2}{3\pi^2}\delta(f)$$

$$+\frac{1}{3\pi^2}\left[\delta\left(f-\frac{2}{T}\right)+\delta\left(f+\frac{2}{T}\right)\right]$$

$$=\left(\frac{3}{4}+\frac{2}{3\pi^2}\right)\delta(f)+\frac{4}{3}\left(\frac{1}{\pi}+\frac{2}{\pi^2}\right)\left(\delta\left(f-\frac{1}{T}\right)+\delta\left(f+\frac{1}{T}\right)\right)$$

$$+\left(\frac{2}{3\pi^2}+\frac{3}{4}\right)\left(\delta\left(f-\frac{2}{T}\right)+\delta\left(f+\frac{2}{T}\right)\right)$$

$$z(t)=\left(\frac{3}{4}+\frac{2}{3\pi^2}\right)+\frac{8}{3}\left(\frac{1}{\pi}+\frac{2}{\pi^2}\right)\cos\left(\frac{2\pi t}{T}\right)+$$

$$+2\left(\frac{2}{3\pi^2}+\frac{3}{4}\right)\cos\left(\frac{4\pi t}{T}\right)$$

---

Es. 2 – Sia data la seguente equaz. alle differenze:

$$9y(n)=7y(n-1)+\frac{8}{9}y(n-2)+18x(n)-7x(n-1)$$

1) Si calcoli la funz. di trasferimento del sistema caratterizzato da questa eq. alle differ.

2) Si trovino tutte le possibili risposte impulsive e le relative zone di convergenza della funzione di trasferimento del pt. 1

3) Si trovi il modulo della risposta in frequenza del filtro a risposta impulsiva causale.

1) $9Y(z) - 7z^{-1}Y(z) - \frac{8}{9}Y(z)z^{-2} = 18X(z) - 7z^{-1}X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{18 - 7z^{-1}}{9 - 7z^{-1} - \frac{8}{9}z^{-2}} = \frac{18z^2 - 7z}{9z^2 - 7z - \frac{8}{9}}$$

$$= \frac{2z\left(z - \frac{7}{18}\right)}{\left(z - \frac{8}{9}\right)\left(z + \frac{1}{9}\right)}$$

2) Il sistema ha 2 poli $p_1 = \frac{8}{9}$, $p_2 = -\frac{1}{9}$

Scriviamo la $H(z)$ per mezzo dei fratti parziali

$$\frac{H(z)}{z} = \frac{A_1}{z - \frac{8}{9}} + \frac{A_2}{z + \frac{1}{9}}$$

$$A_1 = 2\left(z - \frac{7}{18}\right) \cdot \frac{1}{z + \frac{1}{9}}\Bigg|_{z = \frac{8}{9}} = 1$$

$$A_2 = 2\left(z - \frac{7}{18}\right) \frac{1}{z - \frac{8}{9}}\Bigg|_{z = -\frac{1}{9}} = 1$$

$$\Rightarrow H(z) = \frac{z}{z - \frac{8}{9}} + \frac{z}{z + \frac{1}{9}}$$

Scriviamo tutte le possibili $h(n)$

$|z| < \frac{1}{9}$ $\quad h(n) = -\left(-\frac{1}{9}\right)^n u(-n-1) - \left(\frac{8}{9}\right)^n u(-n-1)$

$\frac{1}{9} < |z| < \frac{8}{9}$ $\quad h(n) = \left(-\frac{1}{9}\right)^n u(n) - \left(\frac{8}{9}\right)^n u(-n-1)$

$|z| > \frac{8}{9}$ $\quad h(n) = \left(-\frac{1}{9}\right)^n u(n) + \left(\frac{8}{9}\right)^n u(n)$

la sequenza causale è la terza.

3) $$\overline{H}(f) = H(z)\Big|_{e^{j2\pi fT}} = \frac{e^{j2\pi fT}\left(2e^{j2\pi fT}-\frac{7}{9}\right)}{\left(e^{j2\pi fT}-\frac{8}{9}\right)\left(e^{j2\pi fT}+\frac{1}{9}\right)}$$

$$|\overline{H}(f)| = \frac{\left|2e^{j2\pi fT}-\frac{7}{9}\right|}{\left|e^{j2\pi fT}-\frac{8}{9}\right|\left|e^{j2\pi fT}+\frac{1}{9}\right|}$$

$$= \frac{\sqrt{\left(2\cos 2\pi fT-\frac{7}{9}\right)^2+4\sin^2 2\pi fT}}{\sqrt{\left[\left(\cos 2\pi fT-\frac{8}{9}\right)^2+\sin^2 2\pi fT\right]\left[\left(\cos 2\pi fT+\frac{1}{9}\right)^2+\sin^2 2\pi fT\right]}}$$

$$= \frac{\sqrt{373-252\cos 2\pi fT}}{\sqrt{(145-144\cos 2\pi fT)(82+18\cos 2\pi fT)}}$$