Coherent Detection in Gaussian Disturbance in Presence of Unknown Parameters

Maria S. GRECO
Dipartimento di Ingegneria dell'Informazione
University of Pisa
Via G. Caruso 16, I-56122, Pisa, Italy
m.greco@iet.unipi.it

Generalized Likelihood Text

The generalized likelihood ratio test is a general procedure for composite testing problems. The basic idea is to compare the best model in class H1 to the best in H0, which is formalized as follows. Then we have:

$$\Lambda_{GLRT}(\mathbf{z}) = \frac{\max\limits_{\mathbf{\Phi_1}} p_{\mathbf{z}|H_1}(\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_K; \mathbf{\Phi_1} | H_1)}{\max\limits_{\mathbf{\Phi}_0} p_{\mathbf{z}|H_0}(\mathbf{z}, \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_K; \mathbf{\Phi}_0 | H_0)}$$

This is tantamount to replace the unknown parameters with their ML estimates.

Deterministic Unknown Complex Amplitude - Case #4

$$\mathbf{s}_t = \beta \mathbf{p}(f_d), \quad \beta \text{ deterministic unknown}, \quad \mathbf{p} \text{ perfectly known}$$

$$\mathbf{z} | H_0 \in \mathcal{CN}(0, \mathbf{R}), \quad \mathbf{z} | H_1 \in \mathcal{CN}(\beta \mathbf{p}, \mathbf{R})$$

$$\Lambda(\mathbf{z};\beta) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1;\beta)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

- The LRT depends on the unknown complex target amplitude, therefore it cannot be implemented.

- This is a **composite hypothesis testing problem**.
- A **uniformly most powerful (UMP)** test does not exist.
- We resort to the **generalized likelihood ratio test (GLRT)** $\rightarrow$ the unknown parameters are replaced by their **maximum likelihood estimates (MLE).**

$$\Lambda_{GLRT}(\mathbf{z}) = \max_{\beta} \Lambda(\mathbf{z};\beta) = \frac{\max_{\beta} p_{\mathbf{z}|H_1}(\mathbf{z}|H_1;\beta)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1;\hat{\beta}_{ML})}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

Deterministic Unknown Complex Amplitude - Case #4

- The GLRT is a suboptimal detector. However, it usually produces good detection performance.

$$\ln \Lambda(\mathbf{z};\beta) = \ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1;\beta)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \mathbf{z}^H \mathbf{R}^{-1}\mathbf{z} - (\mathbf{z}-\beta\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z}-\beta\mathbf{p})$$

$$\hat{\beta}_{ML} = \arg\max_{\beta} \left[\mathbf{z}^H \mathbf{R}^{-1}\mathbf{z} - (\mathbf{z}-\beta\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z}-\beta\mathbf{p}) \right]$$

$$= \arg\min_{\beta} \left[(\mathbf{z}-\beta\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z}-\beta\mathbf{p}) \right]$$

$$(\mathbf{z}-\beta\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z}-\beta\mathbf{p}) = \mathbf{z}^H \mathbf{R}^{-1}\mathbf{z} + |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p} - 2\Re e\left\{ \beta \mathbf{z}^H \mathbf{R}^{-1}\mathbf{p} \right\}$$

$$= \mathbf{z}^H \mathbf{R}^{-1}\mathbf{z} + \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p} \cdot \left| \beta - \frac{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}} \right|^2 - \frac{\left| \mathbf{p}^H \mathbf{R}^{-1}\mathbf{z} \right|^2}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}$$

- The minimum is clearly attained when the positive factor containing β is made to vanish.

Deterministic Unknown Complex Amplitude - Case #4

$$\hat{\beta}_{ML} = \frac{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}$$

$$\hat{\beta}_{ML} \equiv \frac{\mathbf{p}^H \mathbf{R}^{-1}(\beta\mathbf{p}+\mathbf{d})}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}} = \beta + \underbrace{\frac{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}}_{\text{estimation error}}$$

$$E\left\{\beta - \hat{\beta}_{ML}\right\} = E\left\{-\frac{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}\right\} = -\frac{\mathbf{p}^H \mathbf{R}^{-1}E\{\mathbf{d}\}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}} = 0$$

$$E\left\{\left|\beta - \hat{\beta}_{ML}\right|^2\right\} = E\left\{\left|\frac{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}\right|^2\right\} = \frac{1}{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\right|^2} E\left\{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H \mathbf{R}^{-1}\mathbf{p}\right\}$$

$$= \frac{\mathbf{p}^H \mathbf{R}^{-1}E\left\{\mathbf{d}\mathbf{d}^H\right\}\mathbf{R}^{-1}\mathbf{p}}{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\right|^2} = \frac{1}{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}}$$

- The MLE of β is unbiased and efficient: the MSE on the estimates of amplitude and phase coincide with their **Cramér-Rao lower bounds (CLRBs)**.

Deterministic Unknown Complex Amplitude - Case #4

$$(\mathbf{z}-\hat{\beta}_{ML}\mathbf{p})^H\mathbf{R}^{-1}(\mathbf{z}-\hat{\beta}_{ML}\mathbf{p})=\mathbf{z}^H\mathbf{R}^{-1}\mathbf{z}-\frac{\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}}$$

- As a consequence we find the GLRT to be:

$$\ln\Lambda_{GLRT}(\mathbf{z})=\ln\frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1;\hat{\beta}_{ML})}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}=\frac{\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

- Incorporating the denominator in the threshold, again we find that the decision strategy is to compare the modulo-squared WMF output to a threshold:

$$\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

Adaptive detection in Gaussian disturbance

- The Optimum and Suboptimum Detectors in previous section have been obtained supposing that the disturbance covariance matrix is a priori known. Most often this is not true and it must be estimated using K secondary data surrounding the CUT.

  - We suppose homogeneous environment:

$\mathbf{z}|H_0$ and $\{\mathbf{z}_k\}_{k=1}^{K}$ are independent and identically distributed (IID)

$$\mathbf{R} = \sigma^2 \mathbf{M} \triangleq E\{\mathbf{z}\mathbf{z}^H | H_0\} = E\{\mathbf{z}_k \mathbf{z}_k^H\}, \quad k = 1, 2, \cdots, K$$

isturbance Covariance Matrix Estimation

- The so-called Sample Covariance Matrix (SCM) estimate is obtained by simply replacing statistical averaging with the secondary data vectors sample mean:

$$\hat{\mathbf{R}} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H$$

- Property: $\det\left(\hat{\mathbf{R}}\right) \neq 0$ with probability 1 if $K \geq N$

isturbance Covariance Matrix Estimation

- It can be proved that it is also the maximum likelihood (ML) estimate if the disturbance is Gaussian distributed.

$$p_{\mathbf{z}}(\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\mathbf{R}) = \prod_{k=1}^{K} p_{\mathbf{z}_k}(\mathbf{z}_k;\mathbf{R}) = \prod_{k=1}^{K} \frac{1}{\pi^N |\mathbf{R}|} \exp\left(-\mathbf{z}_k^H \mathbf{R}^{-1} \mathbf{z}_k\right)$$

$$= \prod_{k=1}^{K} \frac{1}{\pi^N |\mathbf{R}|} \exp\left(-Tr\left\{\mathbf{R}^{-1}\mathbf{z}_k\mathbf{z}_k^H\right\}\right)$$

$$= \frac{1}{\pi^{KN} |\mathbf{R}|^K} \exp\left(-\sum_{k=1}^{K} Tr\left\{\mathbf{R}^{-1}\mathbf{z}_k\mathbf{z}_k^H\right\}\right)$$

$$= \frac{1}{\pi^{KN} |\mathbf{R}|^K} \exp\left(-Tr\left\{\mathbf{R}^{-1}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H\right\}\right)$$

$$= \frac{1}{\pi^{KN} |\mathbf{R}|^K} \exp\left(-Tr\left\{\mathbf{R}^{-1}\mathbf{S}\right\}\right)$$

where $\mathbf{S} \triangleq \sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H$

isturbance Covariance Matrix Estimation

$$\hat{\mathbf{R}}_{ML} = \arg\max_{\mathbf{R}} p_{\mathbf{z}}(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_K; \mathbf{R}) = \arg\max_{\mathbf{R}} \ln p_{\mathbf{z}}(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_K; \mathbf{R})$$

$$= \arg\min_{\mathbf{R}} \left\{ K \ln|\mathbf{R}| + Tr\left\{\mathbf{R}^{-1}\mathbf{S}\right\}\right\} = \arg\min_{\mathbf{R}} \left\{ \ln|\mathbf{R}| + Tr\left\{\mathbf{R}^{-1}\left(\frac{1}{K}\mathbf{S}\right)\right\}\right\}$$

$$= \frac{1}{K}\mathbf{S} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H$$

This estimate is **unbiased** and **consistent**.

$$E\left\{\hat{\mathbf{R}}\right\} = \frac{1}{K}\sum_{k=1}^{K} E\left\{\mathbf{z}_k\mathbf{z}_k^H\right\} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{R} = \mathbf{R}$$

$$\lim_{K\to\infty} \hat{\mathbf{R}} = \mathbf{R} \quad \text{(convergence in mean square sense)}$$

$$i.e. \quad \lim_{K\to\infty} E\left\{\left|\mathbf{R}_{i,k} - \hat{\mathbf{R}}_{i,k}\right|^2\right\} = 0, \quad 1 \le i, k \le N$$

The Adaptive Matched Filter (AMF)

- If we plug the SCM estimate in place of the true one in the WMF detector, we get the so-called Adaptive Matched Filter (AMF):

$$\Lambda_{AMF}(\mathbf{z}) = \Lambda_{WMF}(\mathbf{z})\Big|_{\mathbf{R}=\hat{\mathbf{R}}} = \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- Note that now the denominator is no more an unessential (non data-dependent) scaling factor. Now it depends on the secondary data.

$\mathbf{w} = \hat{\mathbf{R}}^{-1}\mathbf{p}$ data-dependent (adaptive) weights

[Rob92] F. C. Robey, D. L. Fuhrman, E. J. Kelly, and R. Nitzberg, "A CFAR Adaptive Matched Filter Detector," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 29, No.1, pp. 208-216, January 1992.

e Adaptive Normalized Matched Filter (ANMF)

- If we plug the SCM estimate in place of the true one in the NMF detector, we get the so-called Adaptive Normalized Matched Filter (ANMF), a.k.a. Adaptive Coherence Estimator (ACE):

$$\Lambda_{ANMF}(\mathbf{z}) = \Lambda_{NMF}(\mathbf{z})\Big|_{\mathbf{M}=\hat{\mathbf{M}}} = \frac{\left|\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{z}\right|^2}{(\mathbf{z}^H \hat{\mathbf{M}}^{-1}\mathbf{z})(\mathbf{p}^H \hat{\mathbf{M}}^{-1}\mathbf{p})}$$

$$= \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{(\mathbf{z}^H \hat{\mathbf{R}}^{-1}\mathbf{z})(\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p})} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \qquad \text{where } \hat{\mathbf{R}} = \hat{\sigma}^2 \hat{\mathbf{M}}$$

- The statistic of the test is a measure of the similarity (coherence) between the received data vector and the hypothesized target signal vector, that's why it is also called the Adaptive Coherence Estimator (ACE).

Kelly's GLRT

- The binary hypotheses testing problem is stated as follows:

$$\begin{cases} H_0 : \text{Target absent} & \begin{cases} \mathbf{z} = \mathbf{d} \\ \mathbf{z}_k = \mathbf{d}_k, \; k = 1, 2, \cdots, K \end{cases} \\ H_1 : \text{Target present} & \begin{cases} \mathbf{z} = \beta \mathbf{p} + \mathbf{d} \\ \mathbf{z}_k = \mathbf{d}_k, \; k = 1, 2, \cdots, K \end{cases} \end{cases}$$

- This is a composite hypotheses testing problem since some parameters are unknown, i.e. the target complex amplitude b and the disturbance covariance matrix R.

- For this problem a uniformly most powerful test (UMP) does not exist (an UMP test is a rule that maximizes the PD regardless of the unknown parameters of the distribution of the data under H_1, for a preassigned P_{FA}). For this reason we have to resort to the Generalized Likelihood Ratio Test (GLRT).

Kelly's GLRT

- According to the GLRT approach, we have to calculate the likelihood ratio test (LRT), i.e. the ratio of the PDFs of the data vector under the two hypotheses, and then replace the unknown parameters in each PDF by their Maximum Likelihood (ML) estimates:

$$\Lambda_{GLRT}(\mathbf{z}) = \frac{\max\limits_{\beta,\mathbf{R}} p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\beta,\mathbf{R}\,|H_1)}{\max\limits_{\mathbf{R}} p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\mathbf{R}\,|H_0)} = \frac{p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\hat{\beta}_{ML},\hat{\mathbf{R}}_{ML1}\,|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\hat{\mathbf{R}}_{ML0}\,|H_0)}$$

where:

$$(\hat{\beta}_{ML},\hat{\mathbf{R}}_{ML1}) = \arg\max_{\beta,\mathbf{R}} p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\beta,\mathbf{R}\,|H_1)$$

$$\hat{\mathbf{R}}_{ML0} = \arg\max_{\mathbf{R}} p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\mathbf{R}\,|H_0)$$

Kelly's GLRT

- The **maximum likelihood (ML) estimate** of the disturbance covariance matrix under the null hypothesis (H_0):

$$p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\mathbf{R}|H_0) = p_{\mathbf{z}|H_0}(\mathbf{z};\mathbf{R}|H_0)\prod_{k=1}^{K} p_{\mathbf{z}_k}(\mathbf{z}_k;\mathbf{R})$$

$$= \frac{1}{\pi^N|\mathbf{R}|}\exp\left(-\mathbf{z}^H\mathbf{R}^{-1}\mathbf{z}\right)\prod_{k=1}^{K}\frac{1}{\pi^N|\mathbf{R}|}\exp\left(-\mathbf{z}_k^H\mathbf{R}^{-1}\mathbf{z}_k\right)$$

$$= \frac{1}{\pi^{(K+1)N}|\mathbf{R}|^{K+1}}\exp\left(-Tr\left\{\mathbf{R}^{-1}\left(\mathbf{z}\mathbf{z}^H+\mathbf{S}\right)\right\}\right)$$

where $\mathbf{S} \triangleq \sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H$

$$\hat{\mathbf{R}}_{ML0} = \arg\max_{\mathbf{R}} p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\mathbf{R}|H_0) = \frac{1}{K+1}\left(\mathbf{z}\mathbf{z}^H+\mathbf{S}\right)$$

- Analogously, the **maximum likelihood (ML) estimate** of the disturbance covariance matrix under the alternative hypothesis (H_1):

$$p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\beta,\mathbf{R}\,|\,H_1) = p_{\mathbf{z}|H_1}(\mathbf{z};\beta,\mathbf{R}\,|\,H_1)\prod_{k=1}^{K} p_{\mathbf{z}_k}(\mathbf{z}_k;\mathbf{R})$$

$$= \frac{1}{\pi^N |\mathbf{R}|}\exp\left(-(\mathbf{z}-\beta\mathbf{p})^H \mathbf{R}^{-1}(\mathbf{z}-\beta\mathbf{p})\right)\prod_{k=1}^{K}\frac{1}{\pi^N |\mathbf{R}|}\exp\left(-\mathbf{z}_k^H \mathbf{R}^{-1}\mathbf{z}_k\right)$$

$$= \frac{1}{\pi^{(K+1)N} |\mathbf{R}|^{K+1}}\exp\left(-Tr\left\{\mathbf{R}^{-1}\left((\mathbf{z}-\beta\mathbf{p})(\mathbf{z}-\beta\mathbf{p})^H + \mathbf{S}\right)\right\}\right)$$

$$\hat{\mathbf{R}}_{ML1} = \arg\max_{\mathbf{R}} p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\beta,\mathbf{R}\,|\,H_1) = \frac{1}{K+1}\left((\mathbf{z}-\beta\mathbf{p})(\mathbf{z}-\beta\mathbf{p})^H + \mathbf{S}\right)$$

- Inserting the two expressions in the likelihood ratio, we come up with the following problem:

$$\Lambda_{GLRT}(\mathbf{z}) = \frac{\max_{\beta} p_{\mathbf{z}|H_1}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\beta,\hat{\mathbf{R}}_{ML1}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z},\mathbf{z}_1,\mathbf{z}_2,\cdots,\mathbf{z}_K;\hat{\mathbf{R}}_{ML0}|H_0)} = \max_{\beta}\left\{\left[\frac{\left|\hat{\mathbf{R}}_{ML0}\right|}{\left|\hat{\mathbf{R}}_{ML1}\right|}\right]^{K+1}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- which is equivalent to:

$$\Lambda_{GLRT}(\mathbf{z}) \equiv \frac{\left|\mathbf{z}\mathbf{z}^H + \mathbf{S}\right|}{\min_{\beta}\left|(\mathbf{z}-\beta\mathbf{p})(\mathbf{z}-\beta\mathbf{p})^H + \mathbf{S}\right|} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- Minimizing the determinant at the denominator we find:

$$\hat{\beta}_{ML} = \frac{\mathbf{p}^H\mathbf{S}^{-1}\mathbf{z}}{\mathbf{p}^H\mathbf{S}^{-1}\mathbf{p}} \quad \Rightarrow \quad \Lambda_{GLRT}(\mathbf{z}) \equiv \frac{1+\mathbf{z}^H\mathbf{S}^{-1}\mathbf{z}}{1+\mathbf{z}^H\mathbf{S}^{-1}\mathbf{z} - \dfrac{\left|\mathbf{p}^H\mathbf{S}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H\mathbf{S}^{-1}\mathbf{p}}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Kelly's GLRT

Finally, it can be recast in the well-known form:

$$\Lambda_{GLRT}(\mathbf{z}) = \frac{\left|\mathbf{p}^H \mathbf{S}^{-1}\mathbf{z}\right|^2}{(\mathbf{p}^H \mathbf{S}^{-1}\mathbf{p})(1+\mathbf{z}^H \mathbf{S}^{-1}\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

or making explicit the dependence on the SCM:

$$\hat{\mathbf{R}} = \frac{1}{K}\mathbf{S} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H$$

$$\Lambda_{GLRT}(\mathbf{z}) = \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{(\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p})\left(1+\frac{1}{K}\mathbf{z}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

[Kel86] E. J. Kelly, Adaptive Detection Algorithm," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 22, No.2, pp. 115-127, March 1986.

Kelly's GLRT

- The terms in parentheses at the denominator is computationally intensive for real time systems, as it must be calculated for each new input sample.
- This term tends to unity when K is large, hence Kelly's GLRT and AMF tends to be the same for large K.
- It has been proved that Kelly's GLRT outperforms the AMF for small SINRs, instead for high SINRs the AMF usually outperforms the GLRT (this is a confirmation that the GLRT is not a UMP test).
- The absence of the denominator term causes the AMF to be much more sensitive to signals that would appear in the sidelobes of the adapted antenna pattern (mismatched targets), i.e. the AMF is less selective.
- Analytical expressions of AMF, ANMF and Kelly's detector performance can be found in [Kel86] and [Ban09].

[Ban09] F. Bandiera, D. Orlando and G. Ricci. "Advanced Radar Detection Schemes Under Mismatched Signal Models". *Synthesis Lectures on Signal Processing*, 2009, Vol.4, No.1, pp. 1-105. Morgan & Claypool Publisher.

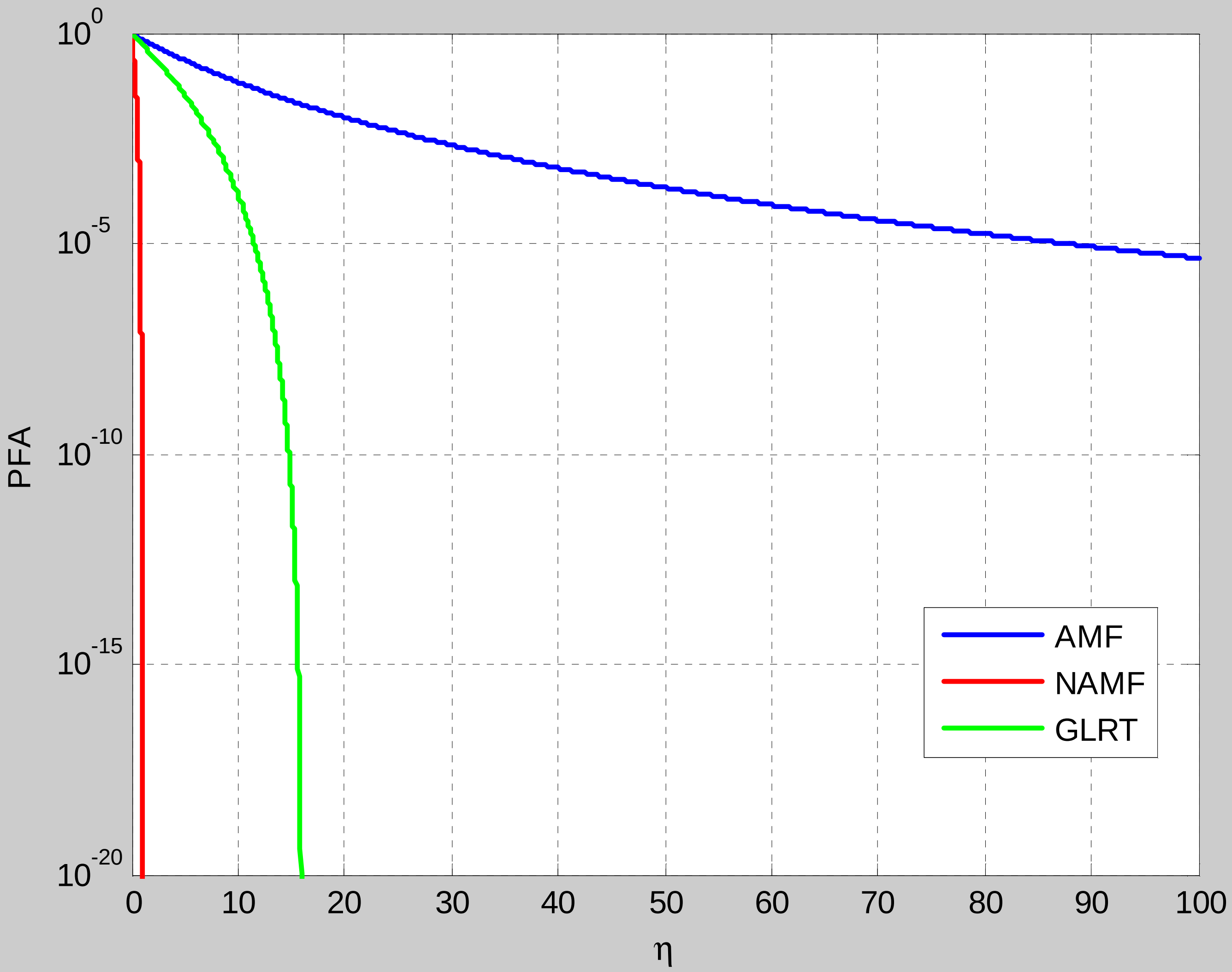
Adaptive detectors

$$\Lambda_{AMF}(\mathbf{z}): \quad \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

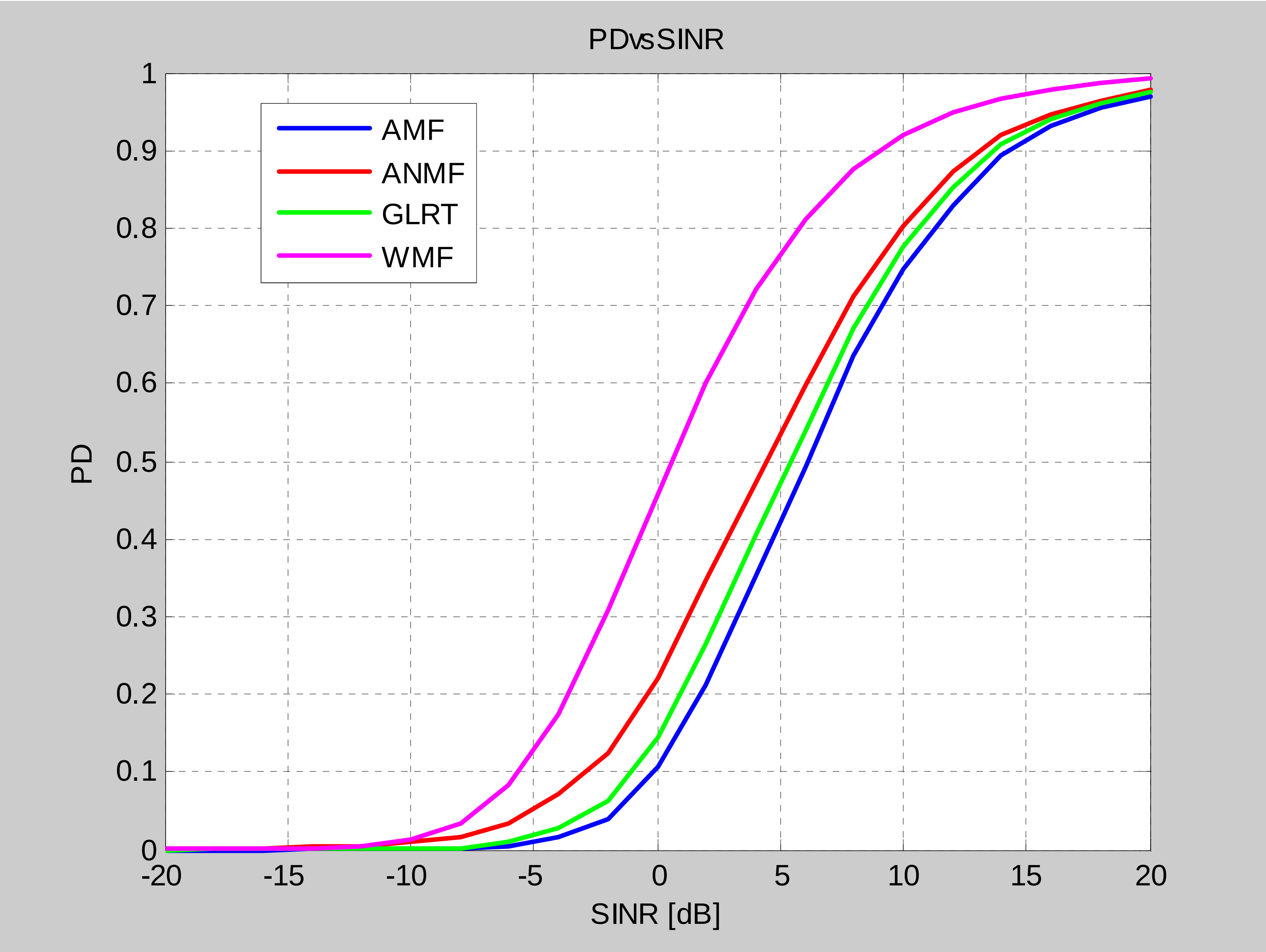
$$\Lambda_{ANMF}(\mathbf{z}): \quad \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \cdot (\mathbf{z}^H \hat{\mathbf{R}}^{-1}\mathbf{z})$$

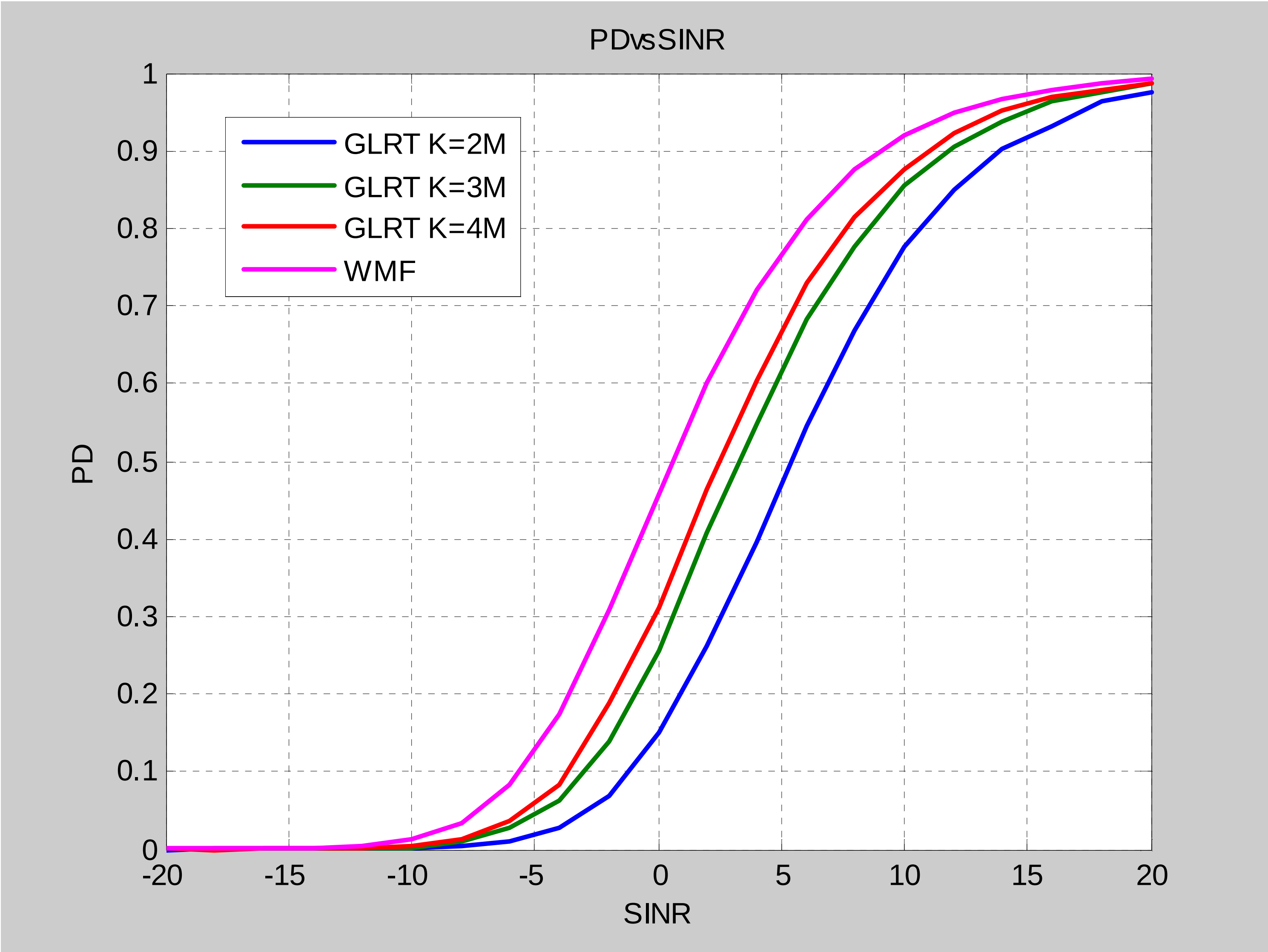
$$\Lambda_{GLRT}(\mathbf{z}): \quad \frac{\left|\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right|^2}{(\mathbf{p}^H \hat{\mathbf{R}}^{-1}\mathbf{p})} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \cdot \left(1 + \frac{1}{K}\mathbf{z}^H \hat{\mathbf{R}}^{-1}\mathbf{z}\right)$$

Adaptive detectors



Adaptive detectors




PDvsSINR
PD
SINR
GLRT K=2M
GLRT K=3M
GLRT K=4M
WMF
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-20
-15
-10
-5
0
5
10
15
20